A New Class of Life Distributions with Multiple Change Points in Failure Rate Function

Technical Report No. ASU/2017/2

Dated: 16 February, 2017

Prajamitra Bhuyan

Applied Statistics Unit,

Indian Statistical Institute,

Kolkata

praja_r@isical.ac.in

Murari Mitra

Department of Mathematics,

Indian Institute of Engineering Science and Technology,

Shibpur

murarimitra@yahoo.com

Anup Dewanji

Applied Statistics Unit,

Indian Statistical Institute,

Kolkata

dewanjia@isical.ac.in

A New Class of Life Distributions with Multiple Change Points in Failure Rate Function

Prajamitra Bhuyan, Murari Mitra and Anup Dewanji

***Abstract*—In many real life scenarios, system reliability depends on dynamic stress-strength interference, where shocks appear at random time points and the strength of a system varies over time. The resulting life distributions exhibit decreasing, increasing, upside-down and bathtub failure rate depending on the nature of strength function. This class of life distributions is useful for reliability analysis of life time data with upside-down and bathtub failure rate. In particular, this new class of distributions can be used to model life time data arising from a distribution with multiple change points in failure rate function. Simulation based algorithm for reliability calculation is proposed and illustrated with numerical examples.**

***Index Terms*—Damage distribution, Inter-arrival time, Numerical integration, Poisson Process, Strength degradation.**

NOTATION

T	Failure time of a system.
T_i	Failure time corresponding to ith the system.
$N(t)$	Shock arrival process.
$F(\cdot)$	Damage distribution.
$F_i(\cdot)$	Damage distribution corresponding to ith the system.
$G(\cdot)$	Inter-arrival time distribution.
$s(t)$	Strength function.
$s_i(t)$	Strength function corresponding to ith the system.
$R(t)$	Reliability function.
$r(t)$	Failure rate function.
$\lambda(t)$	Intensity function of non-homogeneous Poisson process.
$\lambda_i(t)$	Intensity function of non-homogeneous Poisson process corresponding to ith the system.
λ	Intensity of homogeneous Poisson process.

I. INTRODUCTION

Dynamic stress-strength interference can be used to explain wide range of physical and natural phenomena. Reliability analysis based on a mechanistic model provides meaningful insight which can be utilized in product design to improve reliability. Dasgupta and Pecht have provided details of failure mechanisms and corresponding damage models in engineering systems [1]. In many real life scenarios, failure is related to a catastrophic shock, that can be explained by using the non-cumulative damage model, where shocks appear at random time points, causing damage which is only effective at the instant of shock arrival [5], and the system fails if the damage is more than, or equal to, the strength at that time. Fracture of brittle materials such as glasses [2], and semiconductor parts that have failed by some over-current or fault voltage [4, p-21] are real life examples of such models for fixed threshold or strength. Examples such as the impact forces on vehicle wheels due to road bumps, and the forces on building structure due to wind are appropriate real life scenarios for such models [10] with both strength and stress being time dependent. Quite naturally, the resulting life distribution and its properties depends on the strength of the system, shock arrival process and the corresponding damage distribution.

In this paper, we discuss reliability modeling under various generalizations of the Non-cumulative damage model in Section 2. In Section 3, we investigate aging properties of the life distribution arising from the Non-cumulative damage model. In Section 4, we discuss the ordering properties of two different systems. In Section 5, computational issues are discussed and illustrated with numerical examples and in Section 6, we discuss some generalization of shock arrival process and corresponding damage distribution.

P. Bhuyan and A Dewanji are with the Applied Statistics Unit, Indian Statistical Institute, Kolkata, India.

M. Mitra is with Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, India.

II. Non-cumulative Damage Model

Failure time, for the non-cumulative damage model, is the arrival time of the first such shock, when the corresponding damage equals or exceeds the strength at that time. Let $N(t)$ denote the point process representing the number of shocks arriving by time t. Also, the damages due to successive shocks, assume independent of $N(t)$, are denoted by $X_1, X_2, X_3, \ldots$. The reliability function $R(t)$ at time t is then formally defined as

$$\begin{aligned} R(t) &= P[T > t] \\ &= P[X_i < s(\tau_i), \text{ for } i = 1, \ldots, N(t)], \end{aligned} \quad (1)$$

where $\tau_1, \tau_2 \ldots$ denote the successive shock arrival times, T denotes the failure time and $s(t)$ is the strength at time t, satisfying $s(t) \geq 0$ to be continuous. Note that failure occurs only at the shock arrival times. We first consider the successive damages $X_1, X_2, \ldots$ to be independent and identically distributed (iid) following the common distribution $F(\cdot)$ with $F(0) = 0$.

There may be systems which are subject to shocks arriving at prefixed times $0 < \tau_1 < \tau_2 < \ldots$ due to such discrete nature of functioning. This leads to a discrete distribution of T as given by

$$P[T = \tau_i] = [1 - F(s(\tau_i))] \prod_{l<i} F(s(\tau_l)), \quad \text{for } i = 1, 2, \ldots, .$$

and

$$R(t) = \prod_{l \leq i} F(s(\tau_l)), \quad \text{if } \tau_i \leq t < \tau_{i+1},$$

and $R(t) = 1$ for $t < \tau_1$. It can be easily seen that this $R(t)$ satisfies the four properties, namely, (i) $R(0) = 1$, (ii) $\lim_{t\to\infty} R(t) = 0$, (iii) $R(t)$ is non-increasing in t and (iv) $R(t)$ is right-continuous. As a special case, the failure time T follows a geometric distribution with parameter $1 - F(s)$, where $s(\tau_i) = s$, for all $i = 1, 2, \ldots$. When the successive shocks arrive according to a point process $N(t)$ and $s(t) = s > 0$ for all $t \geq 0$, then the reliability function reduces to (See [5])

$$R(t) = \sum_{n=0}^{\infty} P\left[N(t) = n\right]\left[F(s)\right]^n .$$

Xue and Young [10] assumed that $N(t)$ is a non-homogeneous Poisson process (NHPP) with intensity $\lambda(t)$, and the strength $s(t)$ is a decreasing function of t. Then, hypothetically, the successive shock arrivals resulting in a failure follows a filtered Poisson process $N^*(t)$ [7, ch-5], which turns out to be another NHPP with rate $\lambda(t)[1 - F(s(t))]$. Therefore, the reliability function is given by

$$\begin{aligned} R(t) &= P[N^*(t) = 0] \\ &= \exp\{-\int_0^t \lambda(\tau)\{1 - F(s(\tau))\}d\tau\}. \end{aligned} \tag{2}$$

In the special case, when $N(t)$ is a homogeneous Poisson process (HPP) with rate λ, the reliability function reduces to

$$R(t) = \exp\{-\lambda \int_0^t \{1 - F(s(\tau))\}d\tau\}, \tag{3}$$

with the corresponding failure rate $r(t) = \lambda[1 - F(s(t))]$. As a special case, assuming $s(t) = s$ for all $t \geq 0$, the failure time T follows an Exponential distribution with mean $(\lambda\left[1 - F(s)\right])^{-1}$.

In order to study the properties of the above mentioned reliability model, we note, from (2), that the failure rate for corresponding life distribution is given by $r(t) = \lambda(t)\{1 - F(s(t))\}$. We now provide one sufficient condition such that the definition (2) is well-defined. We assume that $0 \leq s(t) \leq M$ for all $t \geq 0$, with $F(M) < 1$ for some positive constant $M > 0$. Note that

$$\begin{aligned} \lambda(t)\{1 - F(s(t))\} &\geq \lambda(t)\{1 - F(M)\} \\ \Longrightarrow \int_0^{\infty} \lambda(t)\{1 - F(s(t))\}dt &\geq \{1 - F(M)\}\int_0^{\infty} \lambda(t)dt \\ &= \infty \end{aligned}$$

Therefore, the failure rate $r(t)$, hence the reliability $R(t)$, given by (2), is well-defined. Therefore, contrary to the common perception of a decreasing strength curve, $s(t)$ does not need to be a decreasing function of t. It is easy to see that the reliability function (2) satisfies the four properties mentioned before. Note that (2) is not well-defined if $s(t)$ is unbounded. For example, $\lim_{t\to\infty} R(t) \neq 0$, if $F(\cdot)$ is Exponential and $s(t) = t$, for all $t > 0$.

III. Models with Specific Ageing

Let us consider the special model (3) with failure rate $r(t) = \lambda[1 - F(s(t))]$ for further investigation. Note that, if $s(t)$ is a decreasing function of t, $r(t)$ is an increasing function of t. This interesting fact prompts us to exploit the flexibility in $s(t)$ to provide a class of life distributions with decreasing, increasing, upside-down and bathtub failure rate depending on the nature of the strength function $s(t)$. Note that $r'(t) = -s'(t)\lambda f(s(t))$, where $f(\cdot)$ is the density function corresponding to the distribution function $F(\cdot)$. Assuming $F(\cdot)$ to be strictly increasing, $f(s(t))$ is positive, and the sign of $r'(t)$ depends on the sign of $s'(t)$. Therefore, the failure rate function of the corresponding life distribution is increasing (decreasing) if $s(t)$ is decreasing (increasing). For example, if $F(\cdot)$ is assumed to be Exponential with mean 1, and $s(t) = 1 + b^t$ (or, $s(t) = 1 - b^t$), for $0 < b < 1$, then $r(t) = \lambda e^{-s(t)}$ is increasing (decreasing). If $b = 0$, then $r(t)$ is a constant (that is, the life distribution is Exponential with rate λe^{-1}). Figure 1 gives the plots of $r(t)$ for these three patterns.

Similarly, if $s(t)$ is upside-down (or bathtub) (e.g. $s(t) = 1 + b^t - c^t$, $0 \leq b, c \leq 1$), then the failure rate function is bathtub (or upside-down) (See Figure 2). One can choose $s(t)$ such that $s'(t) > (<)0$ for all $t_1 < t < t_2$ and $s'(t) < (>)0$ for all $t < t_1$ or $t > t_2$, in order to model life time data arising from a distribution with multiple change points (t_1 and t_2) in the failure rate function. For example, let us consider $s(t) = \frac{0.4}{\sqrt{2\pi}} \exp\{-\frac{(t-5)^2}{2}\} + \frac{0.6}{\sqrt{2\pi}} \exp\{-\frac{(t-10)^2}{2}\}$, $\lambda = 1$, and $F(\cdot)$ to be Exponential with scale parameter unity, as before, leading to three change points in the failure rate function (See Figure 3).

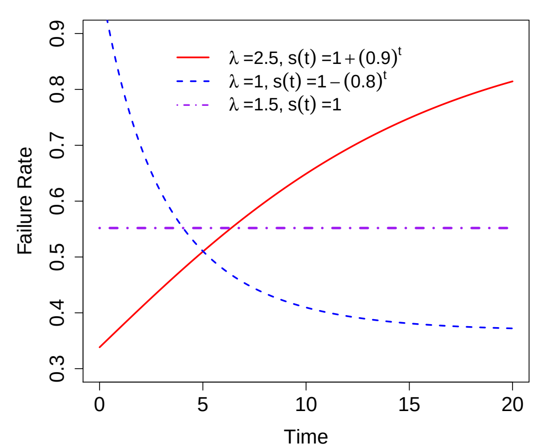

Fig. 1: Increasing, decreasing and constant failure rates

Besides these flexibilities, the failure rate $r(t) = \lambda(t)[1 - F(s(t))]$ corresponding to reliability function (2) reduces to that of any standard life distribution (e.g. Weibull, Gamma, Log-normal, etc.) by taking $s(t) = 0$ for all t, and $\lambda(t)$ as the failure rate of that life time distributions. Therefore, clearly, the class of models defined by $\lambda(t)[1 - F(s(t))]$, with reliability function (2), is a wide class of models encompassing different types of aging phenomena. As mentioned before, the reliability function is well-defined as long as the strength function $s(t)$ is bounded.

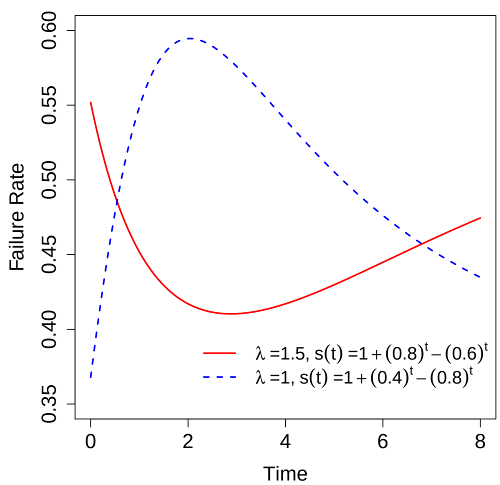

Fig. 2: Bathtub and upside-down failure rates

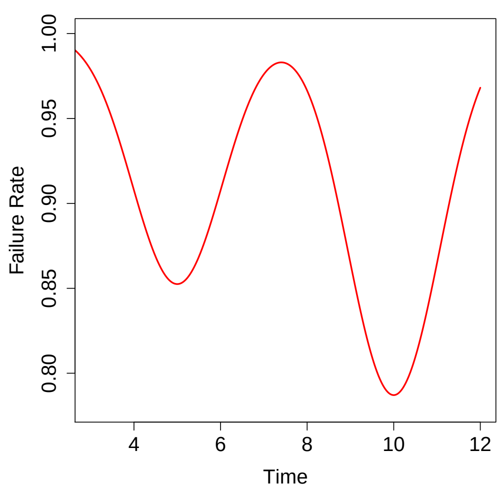

Fig. 3: Failure rate function with $s(t) = \frac{0.4}{\sqrt{2\pi}} \exp\{-\frac{(t-5)^2}{2}\} + \frac{0.6}{\sqrt{2\pi}} \exp\{-\frac{(t-10)^2}{2}\}$ and $\lambda = 1$ having three change points

IV. Ordering Properties

Engineers and industry practitioners are interested to know the ordering properties of the life distributions arising from stress-strength interference. Ordering properties may help to find meaningful insights of the stress-strength mechanism, which can be used for the improvement of the product design.

Let us consider two different systems with life times denoted by T_1 and T_2, respectively, with distributions given by the general form in (2). The damage distribution, the intensity of shock arrival Poisson process, and the strength function are denoted by $F_i(\cdot)$, $\lambda_i(t)$, and $s_i(t)$, respectively, corresponding to the ith system, for $i = 1, 2$. It is easy to verify that T_1 is less than T_2 in stochastic order (i.e. $P[T_1 > t] \le P[T_2 > t]$ for all $t \ge 0$), if $s_1(t) \le s_2(t)$, for all $t \ge 0$, with $F_1(\cdot) \equiv F_2(\cdot)$ and $\lambda_1(\cdot) \equiv \lambda_2(\cdot)$. The same stochastic order holds also when $F_1(x) \le F_2(x)$ for all $x \ge 0$, with $s_1(\cdot) \equiv s_2(\cdot)$ and $\lambda_1(\cdot) \equiv \lambda_2(\cdot)$. Again, this stochastic order holds when $\lambda_1(t) \ge \lambda_2(t)$ for all $t \ge 0$ with $F_1(\cdot) \equiv F_2(\cdot)$ and $s_1(\cdot) \equiv s_2(\cdot)$. Clearly, the reverse stochastic order holds when the corresponding inequalities are reversed.

If $s_1(\cdot) \equiv s_2(\cdot) = s(\cdot)$, say, and $\lambda_1(\cdot) \equiv \lambda_2(\cdot) = \lambda(\cdot)$, say, but the damage distribution of the two systems are stochastically ordered with $F_1(x) \ge F_2(x)$, for all $x \ge 0$, then

$$\frac{P[T_1 > t]}{P[T_2 > t]} = \exp[-\int_0^t \lambda(u)\{F_2(s(u)) - F_1(s(u))\}du]$$

is a non-decreasing function in t; that is, T_1 is greater than T_2 in failure rate ordering. In addition, if we assume that $s(\cdot)$ is non-increasing and the damage distribution of the two systems are ordered in failure rate (that is, $\frac{1-F_1(x)}{1-F_2(x)}$ is non-increasing in x), it can be easily verified that T_1 is greater than T_2 in likelihood ratio .

V. Reliability Computation

Note that, in many situations, the reliability function does not possess closed form expression when both stress and strength are time dependent. In particular, the reliability function (1) cannot be expressed analytically when shock arriving process $N(t)$ is a non-Poisson point process. In such cases, numerical evaluation of reliability is a challenging issue.

In case of Poisson arrival of shocks, the reliability function given by (3) may not possess closed form expression depending upon the choice of $F(\cdot)$ and $s(t)$. Haung and Askin provided algorithm for numerical evaluation of reliability based on Gauss-Legendre quadrature formula considering a decreasing strength function involving random effect parameters [3]. van Noortwijk et al. considered the Gamma process for strength degradation and provided a two step algorithm involving numerical integration and Monte Carlo simulation [9]. However, these algorithms cannot be generalized for the general case where shock arrival process is a non-Poisson point process. In order to address this difficulty, we provide a simple algorithm based on simulation method.

For easy description, let us assume that the successive shocks arrive according to a renewal process $N(t)$ and the successive iid damages $X_1, X_2, \ldots$ are from the common distribution function $F(\cdot)$. Let $U_1, U_2, \ldots$ denote the inter-arrival times between the successive shocks having the common distribution function $G(\cdot)$. The algorithm for simulating a realization of T is provided by the following steps.

$U_0 = 0$, $X_0 = 0$; For $i = 1, 2, \ldots$,
(I) Simulate $U_i \sim G(\cdot)$ and $X_i \sim F(\cdot)$;
(II) Calculate $t_i = \sum_{l=0}^{i} U_l$ and $s(t_i)$;
(III) If $X_i < s(t_i)$, then next i; else, $T = t_i$.

The lifetime distribution is estimated from a large number of, say 10000, realizations of T along with the estimate of reliability $R(t)$ at some chosen points of time.

For illustration, we assume that the shocks arrive according to a Poisson process with rate λ. In this case, inter-arrival distribution is Exponential with mean $1/\lambda$, denoted by $Exp(\lambda)$. We also consider two models for the iid damage distribution F, namely, (i) a Gamma distribution with scale parameter β and shape parameter α, denoted by $Ga(\beta, \alpha)$, with mean being $\alpha\beta$, (ii) a Log-normal distribution with Normal parameters μ and σ, denoted by $LN(\mu, \sigma)$, with mean damage being $\exp(\mu + \frac{1}{2}\sigma^2)$. For our presentation of the results in Table I, for each chosen set of distributional assumptions, we first estimate the pth quantile of the distribution of T by the method of simulation, for $p = 0.1, 0.3, 0.5, 0.7$, and 0.9, as reported in parentheses. The means corresponding to the inter-arrival time distribution and the iid damage distribution are also presented in parentheses. Then, the reliabilities at these quantiles are calculated using numerical integration with 15-point Gauss-Kronrod quadrature formula [6, Ch-2], which are expected to be close to $1 - p$. Computational results for $\lambda = 0.1$, Gamma damage with $\beta = 1$, $\alpha = 5$, and $s(t) = 150 \exp(-0.9t)$ are presented in the top panel of Table I. In the bottom panel of Table I, we present computational results for $\lambda = 0.5$, Log-normal damage with $\mu = 0$, $\sigma = 1$, and $s(t) = 500 \exp(-0.1t)$.

TABLE I: Reliability calculation by numerical integration for iid damages and Poisson arrival of shock with means and estimated quantiles in parentheses.

			$1-p$				
G	F	$s(t)$	0.9	0.7	0.5	0.3	0.1
$Exp(0.1)$	$Ga(1,5)$	$150\exp(-0.9t)$	0.898	0.698	0.502	0.298	0.102
(10)	(5)		(4.967)	(7.490)	(10.794)	(16.004)	(26.713)
$Exp(0.5)$	$LN(0,1)$	$500\exp(-0.1t)$	0.900	0.693	0.497	0.300	0.104
(2)	(1.65)		(45.721)	(51.47)	(55.026)	(58.428)	(63.172)

Note that both the method give similar results under this simple set-up. Our computation indicates that the simulation method has less computational burden than using the numerical integration method. On the other hand, as mentioned before and discussed in detail in the next section, the simulation method has wider domain of applicability even when the numerical integration method fails. Also, one application of the simulation method gives the whole reliability curve, while the other method requires fresh numerical integration for evaluating reliability at each single t. As a by product, one can also obtain the moments of different order from this single application of the simulation method by suitably generalizing the algorithm. Note that both the methods involve some kind of approximation. While the numerical method involves numerical approximation through the use of numerical integration, the simulation method deals with stochastic approximation, in the sense that the resulting reliability is an estimate of the true reliability, through the application of Strong Law of Large Numbers.

VI. Some Generalizations

In many real life scenarios, shocks appear from multiple sources thereby causing damages to follow different distributions depending on the source of the corresponding shock. One can then ideally model the damage distribution corresponding to a shock to have a mixture distribution. In such cases, one can easily compute the reliability $R(t)$ using the above mentioned algorithm.

Sometimes it is of interest for engineers to study life distribution or reliability curve under some stochastic ordering of successive damages coming from the same family of distributions. For example, the successive damages may be stochastically larger. In such situations, the successive damages may be independent but not identically distributed. In general, the non-identical damage distributions need not be from the same family. Also, it is not always realistic to assume the successive damages to be independently distributed [8]. In such case, a multivariate life distribution needs to be used to incorporate inherent dependency among the damages. However, the reliability function (1) does not possess any closed form expression with non-iid or dependent damages. In such cases, one can only use simulation method for numerical evaluation of reliability.

For the purpose of illustration with non-iid damages, we assume that the damage corresponding to the ith shock follows a Gamma distribution with scale parameter $(1.2)^i$ and shape parameter 3. Also, the shocks are assumed to arrive according to a renewal process with distribution of inter-arrival time being $LN(1, 1)$ and strength function $s(t) = 100 \exp(-0.1t)$. The computed reliability curve by using the simulation method is presented in Figure 4 and the estimated first, second and third quartiles are 24.315, 28.428, and 33.522, respectively.

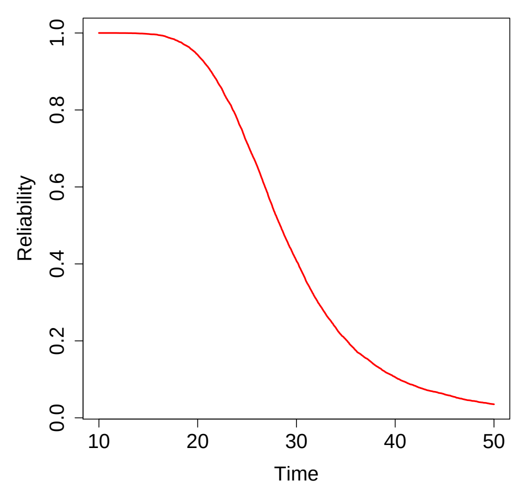

Fig. 4: Reliability Curves with Non-iid Damages and Log-normal Inter-arrival Times

In many real life scenarios, initial strength or its path of deterioration over time is random. Sometimes, deterioration of strength over time is due to various environmental causes changing stochastically at every instant. For the simple scenario with Poisson arrival of shocks, the reliability function (1) involves multiple integrals, and the simulation method appears to have an edge over numerical integration.

REFERENCES

[1] A. Dasgupta and M. Pecht. Material failure mechanisms and damage models. *IEEE Transactions on Reliability*, 40:531536, 1991.

[2] J. D. Esary, A. W. Marshall, and F. Proschan. Shock models and wear processes. *The Annals of Probability*, 1:617–649, 1973.

[3] W. Haung and R. G. Askin. A generalized ssi reliability model considering stochastic loading and strength aging degradation. *IEEE Transactions on Reliability*, 53:77–82, 2004.

[4] T. Nakagawa. *Shock and Damage Models in Reliability Theory*. Springer-Verlag, 2007.

[5] T. Nakagawa and S. Osaki. Some aspects of damage models. *Microelectronics and Reliability*, 13:253–257, 1974.

[6] R. Piessens, E. De Doncker-Kapenga, C. W. Uberhuber, and D. K. Kahaner. *Quadpack a Subroutine Package for Automatic Integration*. Springer-Verlag, 1983.

[7] D. L. Snyder and M. I. Miller. *Random Point Processes in Time and Space*. Springer-Verlag, 1991.

[8] K. Sobczyk and J. Trbicki. Modelling of random fatigue by cumulative jump processes. *Engineering Fracture Mechanics*, 34:477–493, 1989.

[9] J. M. van Noortwijk, J. A. M. van der Weide, M. J. Kallen, and M. D. Pandey. Gamma process and peaks-over-threshold distributions for time-dependent reliability. *Reliability Engineering and System Safety*, 92:16511658, 2007.

[10] J. Xue and K. Yang. Upper and lower bounds of stress-strength interference reliability with random strength degradation. *IEEE Transactions on Reliability*, 46:142–145, 1997.