Robust estimation of directional mean

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Abstract

In this paper, we attempt to address the robustness issues in estimating mean of a circular distribution. Without assuming any specific circular distribution, we first discuss the robustness issue of the estimators in general. Then, we propose a maximum trimmed cosine estimator in this context and discuss the algorithm for its computation. The method is evaluated by detailed simulation study and is illustrated by some real data sets. We discuss some existing competitors of the proposed method. We then extend the method for estimation of spherical mean.

1 Introduction

Circular data or directional data is a relatively unexplored area of statisics despite having numerous applications in meteorolgy, astronomy, geophysics and opthalomology. Mardia and Jupp (2000) provides a detailed discussion on the analysis of circular random variables. There are numerous real data examples in biology, ecology, environment and medical science where the orientation or the mean direction of circular data is of interest. The directions of 100 ants chose to travel when they are exposed to a black target are given in Fisher (1993, p. 243). The scientific question under study was whether or not the ants would run toward homeward direction. Goodyear (1970) illustrated the movement pattern measured under heavily overcast conditions for 50 fish, starhead topminnows. See also Fisher (1993, p. 59) in this connection. Schmidt-Koenig (1963) provided another dataset on the vanishing directions of 15 homing pigeons. Fisher (1993, p. 252) has given data on 31 blue periwinkles, Nodilittorina unifasciata, which are tiny blue snails (about the size of a little fingernail), transplanted downshore from the location, at which they normally live. With reference to these datasets, Otieno and Anderson-Cook (2006a) discussed various descriptive measures like the circular mean, circular variance, circular mean deviation and circular median absolute deviation. Otieno and Anderson-Cook (2006a) pointed out the existence of number of outliers in several real datasets. However, they did not provide any robust descriptive statistics, possibly due to the lack of robust descriptive measure in the literature. The present paper aims to fulfil that gap.

There is inadequate literature available on robust estimation in a compact set. However, there is strong need for the robust estimation for spherical data especially when observations are concentrated toward a certain direction; see Watson (1983) and Kato and Eguchi (2014) in this context. Robust estimators for the mean direction for the circular, or two-dimensional, case were discussed in Mardia (1972, p. 28) and Lenth (1981). Robust estimation of circular mean was considered by Otieno and Anderson-Cook (2006a). Agostinelli (2007) considered robust estimation of the unknown parameters of a circular distribution using weighted likelihood and minimum disparity, where the class of power divergence and the related residual adjustment function is investigated for von Mises and wrapped normal distributions. Fisher (1985), Ducharme and Milasevic (1987), Ko and Guttorp (1988) and Chan and He (1993) discussed the estimation of circular mean for general dimension. The estimation for the concentration parameter was discussed by Fisher (1982), Ducharme and Milasevic (1990), Ko (1992), Laha and Mahesh (2012), among others. Robust estimation of both location and concentration parameters for the von Mises-Fisher distribution is then considered by Kato and Eguchi (2014). They first showed that the maximum likelihood estimator for the von Mises-Fisher distribution is not always robust, and then proposed two families of robust estimators, again minimising two density power divergences.

He and Simpson (1992) advocated the use of the circular median, instead of circular mean, as an estimate of preferred direction, particularly when the data are not from the von Mises distribution. See also Otieno and Anderson-Cook (2003, 2006b) for Hodges-Lehmann type estimators towards alternatives to circular mean as estimate of orientation.

In the present paper, we do not consider any particular distribution. Under a non-parametric setup, we want to provide robust estimator or circular mean, irrespective of the underlying distribution. Let $\theta_{x_1}, \ldots, \theta_{x_n} \in [0, 2\pi)$ are independent circular observations. Then, the circular mean, as defined in Mardia and Jupp (2000) is obtained as the follows. Let $z_1, \ldots, z_n$ are the corresponding unit complex numbers such that $z_i = e^{i\theta_{x_i}}$. Then, the circular mean is defined as

$$\overline{\theta} = \arg(z_1 + \cdots + z_n). \tag{1.1}$$

Our objective in the present paper is to provide robust estimation of circular mean given in (1.1).

The rest of the paper is organized as follows. In Section 2, we introduce *circular breakdown point*, which is the analogue of *breakdown point* applicable for circular data. In Section 3, we discuss the *maximum trimmed cosine estimator* for estimating circular mean. Computational algorithm is also suggested. Simulation study is reported and discussed in Section 4. In Section 5, we extend the approach for finite dimensional unit sphere. Data analyses are done in Section 6. Section 7 concludes with a discussion on robustness issues of *circular variance.*

2 Circular breakdown point

The concept of Breakdown point (BDP) was first introduced in Donoho and Huber (1983). The Breakdown point is defined in terms of the smallest fraction $\frac{m}{n}$ of total observations n which can be contaminated in a way such that the estimator may go arbitrarily far from the estimator based on all the observations. Formally,

$$BDP = \inf \left\{ \frac{m}{n} : \sup ||T(X) - T'(X)|| = \infty \right\}. \tag{2.1}$$

Here $T(X)$ is the estimator based on all the n observations, while $T'(X)$ is the estimator when m out of n observations are contaminated arbitrarily. For example, it is well-known that the BDP of OLS estimator in simple linear regression setup is $\frac{1}{n}$; and the median is much more robust estimator of location, the BDP of median is $\left(\left[\frac{n}{2}\right] + 1\right)$.

The definition (2.1) is not appropriate for circular data, in general. As $\overline{\theta} \in [0, 2\pi)$, if we define the breakdown point of the circular mean in the traditional sense given in (1.1), then it does not have any meaning as the distance between the mean of the original set of observations and the contaminated set of observations will always be finite. Thus, a new definition is necessary for detecting outliers in the model and checking the robustness of the circular mean.

Let, $\Theta = [0, 2\pi)$ represents the whole parameter space and Θ_C represents a proper subset of Θ, then *Circular Breakdown Point* (CBDP) is defined as

$$CBDP = \inf \left\{ \frac{m}{n} : T(X) \notin \Theta_C \right\}, \tag{2.2}$$

where $T(X)$ is the estimator of mean when m out of n observations are arbitrarily contaminated. We shall use this definition to find the CBDP of circular mean.

For $\bar{\theta}$, the standard estimate of circular mean, given by (1.1), we have the following theorem.

Theorem 1. The CBDP of circular mean is $\frac{1}{n}$.

Proof: Let $z_1 + \cdots + z_{n-1} = 0$; then clearly $\arg(T(X)) = \arg(z_n)$. Hence, $T(X)$ can take any value on Θ and thus there does not exist any $\Theta_C \subset \Theta : T(X) \in \Theta_C$. Thus, we get $m = 1$. So, the CBDP for circular mean is $\frac{1}{n}$. □

Thus, we need a more robust estimator for the circular mean having $CBDP > \frac{1}{n}$. We discuss such a robust estimator of the circular mean and its computing algorithm in the next section.

3 Maximum trimmed cosine estimator for circular mean

3.1 Maximum trimmed cosine estimator

In the quest for robust estimators, least absolute values estimator was proposed by Edgeworth (1987) and M-estimator was given by Huber (1973, p. 800). But, in terms of BDP, the problem remained intact. In the pursuit of such a robust estimator with higher breakdown point, Rousseuw (1985) introduced the Least Trimmed Squares (LTS) estimator which has the $BDP = ([n/2] + 1)/n$ for simple linear regression.

It is well known that in the circular data setup where the observations are on the circumference of a unit circle, the residual is best described in terms of the cosine of the difference between the observed and predicted values; the cosine residual is defined as $\cos(\theta_j - \theta_0)$, where θ_j is the observed jth angular response and θ_0 is the estimator for circular mean, $\theta_0 \in [0, 2\pi)$. The higher the value of this cosine residual, the better is the fit. Thus, the Maximum Trimmed Cosine Estimator (MTCE) can be defined as the value of the parameters which maximizes

$$\sum_{j=1}^{h} \cos(\theta_j - \theta_0), \tag{3.1}$$

where $\left[\frac{n}{2}\right] + 1 \leq h \leq n$. Thus, it gives the value of the parameter which fits h observations out of n in the best way. This can be viewed in the same spirit of obtaining the Maximum Trimmed Likelihood Estimator (MTLE) from the classical maximum likelihood estimator (MLE) (c.f. Neykov and Neytchev, 1990; Bednarski and Clarke, 1993; Vandev and Neykov, 1993). MTCE is based on h observations out of the n total observations. Naturally h is taken to be greater than $\left[\frac{n}{2}\right]$ as a lower value of h indicates that more than half of the observations are contaminated which does not make much sense. MTCE can be extended in the context of circular-circular regression. But, in the present paper, we concentrate only on circular mean without regression.

Theorem 2. The MTC estimator maximizing (3.1) has CBDP given by $(\left[\frac{n}{2}\right] + 1)/n$.

Proof: The lower limit of BDP of this estimator can be calculated by Theorem 1 of Vandev and Neykov (1998). Then, if we define $f_j(x_j; \theta_0) = 1 + \cos(\theta_{x_j} - \theta_0)$, then $f_j \geq 0$ and the ordering will remain the same. The f_j's are arranged in decreasing order and the first h out of n f_j's are taken. To find out the CBDP, we have to find the minimum value of d such that the set $\Phi = \{\theta : \frac{1}{d}\sum_{j=1}^{d} f_j(x_j; \theta) \geq C\} \subset \Theta$ for all $C \in (0, 1]$. Now, for $d = 1$, $\cos(\theta_j - \theta_0) \geq k$ implies that θ_0 belongs to closed and proper subset for $k \in (-1, 1]$. Thus, by Theorem 1 of Muller and Neykov (2001), if $h \geq (\left[\frac{n}{2}\right] + 1)$, we have $CBDP \geq \frac{1}{n}\min\{N - h + 1, h\}$, and the maximum value of CBDP is at $h = (\left[\frac{n}{2}\right] + 1)$, when the CBDP becomes $(\left[\frac{n}{2}\right] + 1)/n$. □

3.2 Computation of MTCE for circular mean

The time needed for computation of the MTCE is very large as we have to take all possible $\binom{n}{h}$ subsets of the n observations and then find the subset for which $\sum_{j=1}^{h} \cos(\theta_{(j)} - \theta_0)$ is maximum, where $\theta_{(j)}$ is the jth ordered observation. In this subsection, our objective is to reduce this computational time considerably. The algorithm for reducing the computational time complexity is based on partitioning of parameter set as mentioned in Klouda (2015). We suggest an algorithm assuming that no two observations θ_l, θ_j are the same. In fact, this has probability 0 in case of any continuous distribution. Then, for any two observations x_l and x_j, the expression $\cos(\theta_l - \theta_0) = \cos(\theta_j - \theta_0)$ implies

$$\frac{z_l}{z_0} = \overline{\left(\frac{z_j}{z_0}\right)}. \tag{3.2}$$

Thus, we get $z_0^2 = z_l z_j$, which has two solutions. Now, for each of these solutions we shall order all the observations and check if $\cos(\theta_j) = \cos(\theta_{(h)})$. If this is the case, then we shall take the corresponding weight set and find the sum. We shall do this for all $\binom{n}{2}$ pairs of observations and take the parameter θ_0 to get the MTCE for the circular mean. Denoting any pair of two observations by ν_s, where $1 \leq s \leq \binom{n}{2}$, the algorithm is written below.

Algorithm C:

Set $s = 1$, $\theta_0 = 0$ and $M = -\infty$.

Step 1: Solve (3.2) to get the two solutions for θ_0.

Step 2: For each of these solutions of θ_0, check if the residual corresponding to the chosen observations is the hth ordered residual. The maximum number of the chosen weights can be 3.

Step 3: MTC_{ν_s} = maximum of the sum of residuals.

Step 4: If $MTC_{\nu_s} > MTC_{\max}$, then $MTC_{\max} = MTC_{\nu_s}$ and $\theta_0 = \theta_0(\nu_s)$.

Step 5: If $s < \binom{n}{2}$, then $s = s + 1$. Else, end.

Thus, the complexity of the new algorithm is $6\binom{n}{2}$ which is much less than $\binom{n}{h}$, specially for large n.

4 Simulation study

Simulation studies were carried in the 2-dimensional case for three sample sizes 15, 30 and 50 with mean angle 0 at two different values of κ when the observations follow von Mises distribution and two different values of ρ when the observations follow Wrapped Cauchy distribution. The observations were contaminated with mean direction $\pi/2$ in all the cases with $\kappa = 5$ in the von Mises case and $\rho = 0.9$ in the Wrapped Cauchy case. 10,000 simulations were carried out in all the cases. The comparison of MTCE is shown with circular mean and circular median. The circular median calculated here is as defined in Fisher (1993, p. 36). The axis is taken which divides the data in two equal groups and then the end of this axis along which more observations are concentrated is chosen as circular median. The number of contaminated observations is denoted by ν. Here M_G denotes the number of contaminated observations included in the calculation of MTCE. The circular means over 10,000 simulations for mean, median and MTCE are reported and the circular dispersion $(1 - \bar{R})$, the average resultant length of the unit complex numbers corresponding to all estimates are reported in the parentheses. From Tables 1 and 2, it can be concluded that MTCE performs better than circular mean in all the cases. At low level of contamination, MTCE performs better than the median but at higher contamination, the performance of circular median is comparable to MTCE and it outperforms MTCE in some cases.

5 Maximum trimmed cosine estimator for spherical mean

5.1 Spherical breakdown point

Robust estimation method for circular mean can be extended to the case of spherical observations on the surface of the unit sphere in p dimensions. Let $x_j \in \mathbb{S}^{p-1}$, $j = 1, \ldots, n$. The mean of a spherical observations on the surface of the unit sphere is defined as

$$\bar{x} = \frac{\sum_j^n x_j}{n} \bigg/ \left|\left|\frac{\sum_j^n x_j}{n}\right|\right| ,$$

where $||\cdot||$ is the norm of a vector.

Similar to CBDP, the *spherical breakdown point* (SBDP) of the estimator of the spherical mean can be defined. Let, Θ represents the whole parameter space and Θ_C represents a proper subset of Θ, then SBDP is defined as

$$SBDP = \inf\left\{\frac{m}{n} : T(X) \notin \Theta_C\right\}, \tag{5.1}$$

Table 1: Different types of estimates of the parameters (with standard errors in parentheses) when angular error follows von Mises distribution with parameter κ and $\theta_0 = 0$.

			Estimated Values (s.e.) of			
n	κ	v	Mean	Median	MTCE	M_G
15	5	3	0.244(0.009)	0.147 (0.014)	0.028 (0.031)	0.189 (0.007)
	5	5	0.462 (0.009)	0.303 (0.019)	0.270 (0.017)	1.480 (0.021)
	4	3	0.252 (0.011)	0.164 (0.018)	0.042 (0.042)	0.240 (0.008)
	4	5	0.472 (0.011)	0.338 (0.023)	0.344 (0.129)	1.753 (0.022)
30	5	3	0.108 (0.004)	0.062 (0.006)	-0.002 (0.023)	0.056 (0.004)
	5	6	0.241 (0.007)	0.142 (0.004)	0.031 (0.033)	0.400 (0.013)
	5	9	0.402 (0.004)	0.254 (0.008)	0.181 (0.086)	1.833 (0.033)
	4	3	0.113 (0.005)	0.070 (0.007)	-0.005 (0.030)	0.086 (0.003)
	4	6	0.252 (0.006)	0.164 (0.009)	0.039 (0.044)	0.502 (0.015)
	4	9	0.415 (0.006)	0.289 (0.011)	0.237 (0.110)	2.284 (0.035)
50	5	5	0.111 (0.002)	0.065 (0.004)	-0.008 (0.021)	0.105 (0.006)
	5	10	0.245 (0.003)	0.146 (0.004)	0.032 (0.032)	0.703 (0.022)
	5	15	0.404 (0.003)	0.257 (0.005)	0.205 (0.091)	3.340 (0.055)
	4	5	0.114 (0.003)	0.072 (0.005)	-0.007 (0.027)	0.162 (0.008)
	4	10	0.253 (0.003)	0.165 (0.005)	0.041 (0.042)	0.893 (0.025)
	4	15	0.417 (0.003)	0.290 (0.006)	0.259 (0.114)	4.001 (0.059)

Table 2: Different types of estimates of the parameters (with standard errors in parentheses) when angular error follows Wrapped Cauchy distribution with parameter ρ and $\theta_0 = 0$.

			Estimated Values (s.e.) of			
n	ρ	v	Mean	Median	MTCE	M_G
15	0.9	3	0.245(0.006)	0.048 (0.002)	0.001 (0.008)	0.063 (0.004)
	0.9	5	0.462 (0.007)	0.121 (0.009)	0.095 (0.067)	0.595 (0.015)
	0.75	3	0.291 (0.018)	0.131 (0.018)	0.064 (0.091)	0.361 (0.009)
	0.75	5	0.542 (0.019)	0.373 (0.052)	0.482 (0.232)	2.020 (0.023)
30	0.9	3	0.111 (0.003)	0.019 (0.001)	-0.004 (0.004)	0.027 (0.002)
	0.9	6	0.245 (0.003)	0.044 (0.001)	-0.001 (0.005)	0.102 (0.006)
	0.9	9	0.406 (0.003)	0.087 (0.002)	0.032 (0.025)	0.455 (0.017)
	0.75	3	0.132 (0.008)	0.051 (0.005)	-0.022 (0.030)	0.103 (0.005)
	0.75	6	0.292 (0.009)	0.123 (0.007)	0.047 (0.073)	0.587 (0.016)
	0.75	9	0.474 (0.009)	0.239 (0.015)	0.339 (0.209)	2.726 (0.038)
50	0.9	5	0.111 (0.001)	0.018 (0.003)	-0.004 (0.003)	0.049 (0.003)
	0.9	10	0.245 (0.002)	0.043 (0.001)	-0.001 (0.005)	0.171 (0.010)
	0.9	15	0.406 (0.002)	0.082 (0.001)	0.027 (0.020)	0.691 (0.026)
	0.75	5	0.133 (0.005)	0.050 (0.003)	-0.027 (0.0022)	0.135 (0.006)
	0.75	10	0.291 (0.005)	0.118 (0.004)	0.029 (0.059)	0.855 (0.025)
	0.75	15	0.476 (0.005)	0.230 (0.008)	0.355 (0.209)	4.702 (0.064)

Similar to the the case of circular observations, the SBDP for the spherical mean is $\frac{1}{n}$. To increase the robustness, again the MTC can be similarly defined in this case to be $\sum_{j=1}^{h} y_{(j)}^T x_{(j)}$, such that (j)th observation is the jth order residual in decreasing order. It can be claimed directly, as in the case of circular mean that $d = 1$ and hence, the SBDP in this case when $h = \left[\frac{n}{2}\right] + 1$ is $\left[\frac{n}{2}\right]/n$.

5.1.1 Computation of MTC for spherical mean

Like the case of computation for the MTC in case of circular-circular regression and circular mean, the computational complexity for the MTC in the case of spherical mean can also be reduced a lot by taking the weights at the points where the residuals with respect to some observations are the same at the number of such weights are finite. Then, we do not need to calculate the parameter with respect to all the $\binom{n}{h}$ sets of observations.

Let $x_l, x_j \in \mathbb{S}^{p-1}$ are two observations. Then, $\{z : (x_l - x_j)^T z = 0\}$ gives a hyperplane $\mathbb{R}^{p-1}$. Similarly, taking p observations $y_1, y_2, \ldots y_p$ and solving the equations

$$(y_1 - y_2)^T z = (y_2 - y_3)^T z = (y_p - y_{p-1})^T z = 0 \tag{5.2}$$

gives the solution space of $z \in \mathbb{R}$ passing through origin. Now, the spherical mean has the property that its norm is 1. Hence, we get only two solutions. For each of these two solutions we shall take the weight such that the sum of first h residual is maximum. Then, corresponding to each weight set, the spherical mean can be obtained. Then, the sum of first h residuals with respect to this spherical mean can be calculated. The final estimator will then be the one which has the maximum sum of first h residuals.

Denoting any set of p observations by ν_s, where $1 \leq s \leq \binom{n}{p}$, the algorithm is written below:

Algorithm S:

Set $s = 1$, $z = 0$ and $M = -\infty$.

Step 1: Solve (5.2) to get the two solution for $z \in \mathbb{S}^{p-1}$.

Step 2: For each of these z, check if the residual corresponding to the chosen observations is the hth ordered residual.The maximum number of the chosen weights can be $2^p - 1$.

Step 3: MTC_{ν_s} = maximum of the sum of residuals.

Step 4: If $MTC_{\nu_s} > MTC_{\max}$, then $MTC_{\max} = MTC_{\nu_s}$ and $z = z(\nu_s)$.

Step 5: if $s < \binom{n}{p}$, $s = s + 1$. Else, end.

The complexity of this algorithm is $2(2^p - 1)\binom{n}{p}$ which is less than $\binom{n}{h}$.

5.2 Simulation

Here MTCE is compared with spherical mean and spherical median. The spherical median, as mentioned in Fisher (1985), is that point on the unit sphere from which the sum of arc lengths of the observations is minimum. As earlier 10,000 simulations were carried out for two different sample sizes at two levels of contamination. The mean is chosen as $\mu = (1, 0, 0)$ while the contamination is done with mean as $(0, 1, 0)$ and $\rho = 0.9$ for the exit distribution and $\kappa = 5$ for von Mises-Fisher distribution. In Tables 3 and 4, "Mean" denotes the spherical mean and "Median" denotes the spherical median and MTCE denotes the spherical mean of the MTCE over 10,000 simulations. The dispersion of mean, median and MTCE are denoted by R_1, R_2 and R_3 respectively. It can be seen that the MTCE is closer to the uncontaminated mean than the spherical mean and spherical median in all the cases.

6 Data Analysis

The data analysis example for the case of MTCE in 2-dimensions is considered from two datasets mentioned in Otieno and Anderson-Cook (2006a). In the first dataset, 15 homing pigeons were released some distance away from their loft in the Northwest direction. The direction of loft from the point of release was 149 degrees. It was expected that the dominant direction of the departing flight would be in the direction of the loft. Circular mean, circular median and MTCE were considered for measuring the dominating direction. The value of the circular mean, circular median and MTCE came to be 172.117 degrees, 150 degrees and 148.740 degrees respectively. Hence, as the MTCE is based on the majority of observations only, it shows that the MTCE is very near to 149 degrees and hence, barring the outliers, the preferred direction of the pigeons is in the direction of the loft. The median is also very close to the preferred direction. But, due to the presence of some outliers, circular mean is far from the dominating direction.

Table 3: Different types of estimates of the parameters (with standard errors in parentheses) when angular error follows vonMises Fisher with parameter κ and $\mu = (1, 0, 0)$.

			Estimated Values (s.e.) of						
n	κ	v	Mean	R_1	Median	R_2	MTCE	R_3	M_G
30	5	6	(0.971,0.241,-0.000)	0.011	(0.986,0.168,-0.001)	0.013	$(1, 0.015, 0.001)$	0.027	0.211(0.005)
	5	9	(0.920,0.391,0.002)	0.012	(0.956,0.294,0.002)	0.015	$(0.999, 0.038, -0.001)$	0.030	0.528(0.010)
	4	6	(0.967,0.256,-0.000)	0.014	(0.981,0.194,-0.001)	0.018	$(1, 0.028, -0.001)$	0.037	0.353(0.007)
	4	9	(0.910,0.416,0.000)	0.016	(0.942,0.336,0.001)	0.020	$(0.997, 0.079, -0.003)$	0.050	0.953(0.018)
50	5	10	(0.970,0.241,-0.001)	0.007	(0.986,0.169,-0.002)	0.008	$(1, 0.012, 0.001)$	0.018	0.277(0.006)
	5	15	(0.920,0.392,0.000)	0.007	(0.956,0.295,-0.000)	0.009	$(1, 0.027, -0.000)$	0.018	0.657(0.010)
	4	10	(0.966,0.257,0.000)	0.009	(0.981,0.195,0.000)	0.011	$(1, 0.022, -0.001)$	0.024	0.448(0.008)
	4	15	(0.910,0.414,-0.001)	0.009	(0.942,0.336,-0.002)	0.012	$(0.998, 0.057, -0.001)$	0.031	1.200(0.018)

Table 4: Different types of estimates of the parameters (with standard errors in parentheses) when angular error follows Exit distribution with parameter ρ and $\mu = (1, 0, 0)$.

			Estimated Values (s.e.) of						
n	ρ	v	Mean	R_1	Median	R_2	MTCE	R_3	M_G
30	0.9	6	(0.970,0.243,0.001)	0.004	(0.999,0.049,-0.000)	0.001	$(1, -0.001, -0.000)$	0.001	0.007(0.000)
	0.9	9	(0.919,0.393,-0.000)	0.004	(0.995,0.096,-0.001)	0.002	$(1, 0.003, 0.000)$	0.001	0.023(0.010)
	0.75	6	(0.958,0.286,0.000)	0.011	(0.991,0.133,0.001)	0.008	$(1, 0.000, -0.000)$	0.008	0.058(0.009)
	0.75	9	(0.889,0.458,0.000)	0.012	(0.967,0.254,-0.001)	0.002	$(0.999, 0.016, -0.001)$	0.018	0.275(0.010)
50	0.9	10	(0.970,0.243,-0.000)	0.002	(0.999,0.049,-0.000)	0.001	$(1, 0.000, 0.000)$	0.001	0.011(0.007)
	0.9	15	(0.919,0.394,-0.001)	0.003	(0.996,0.093,-0.000)	0.001	$(1, 0.003, -0.000)$	0.001	0.037(0.011)
	0.75	10	(0.960,0.279,0.001)	0.007	(0.992,0.130,-0.000)	0.005	$(1, 0.001, 0.001)$	0.005	0.079(0.010)
	0.75	15	(0.895,0.446,0.001)	0.007	(0.969,0.249,0.001)	0.007	$(1, 0.005, 0.000)$	0.007	0.237(0.006)

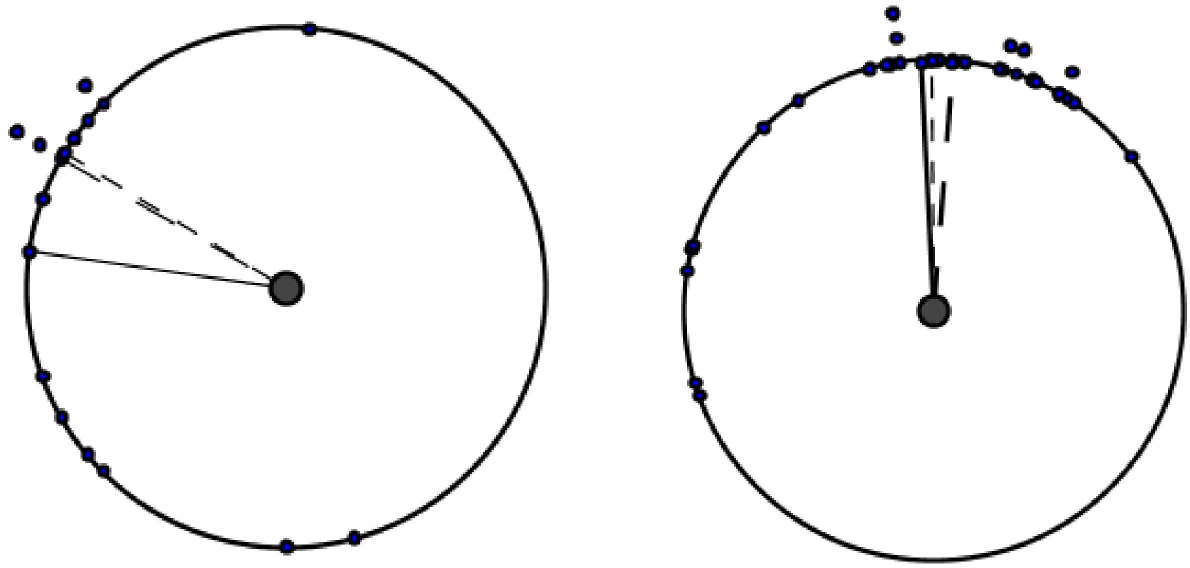

Figure 1: Solid line shows the circular mean, small dotted line shows the MTCE and big dotted line shows the circular median direction for (Left): Homing Pigeon dataset and (Right:) Periwinkles dataet.

In the next example, the direction of movement of 31 blue periwinkles as mentioned in Otieno and Anderson-Cook (2006a) is considered. The rose diagram (not shown here) indicates that the directions are more or less symmetrically distributed about the mean direction. The values for circular mean, circular median and MTCE for this dataset are 92.819, 86.001 and 90.585 degrees respectively. Hence, due to the symmetry about the mean direction, all the three measures are close to each other in this case. Figure 1 shows the directions of the pigeons and periwinkles. The circular mean, circular median and MTCE are also shown by different radii in the figure.

The vectorcardiogram dataset mentioned in Downs(2003) is considered for the example of MTCE in the case of spherical data in 3-dimensions. Two vectorcardiograms, one using the Frank system and the other using the McFee system, were taken from 28 boys and 25 girls. The readings corresponding to Frank system were denoted by $F_{\max} = (F_x, F_y, F_t)^\top$ and the readings corresponding to McFee system were denoted by $M_{\max} = (M_x, M_y, M_t)^\top$. Here $F_{\max}$ and $M_{\max}$ are unit vectors in the direction of maximum QRS loop vector. Four cases are considered, 2 for the data related to the boys and 2 for the data related to the girls

Table 5: Data Analysis for spherical case

	Mean	Median	MTCE
		Frank System	
Boys	(0.495,0.639,0.589)	(0.503,0.656,0.563)	(0.468,0.657,0.591)
Girls	(0.441,0.698,0.564)	(0.447,0.691,0.569)	(0.432,0.701,0.567)
		McFee System	
Boys	(0.666,0.705,0.244)	(0.700,0.642,0.313)	(0.695,0.646,0.316)
Girls	(0.648,0.663,0.376)	(0.675,0.643,0.363)	(0.701,0.656,0.280)

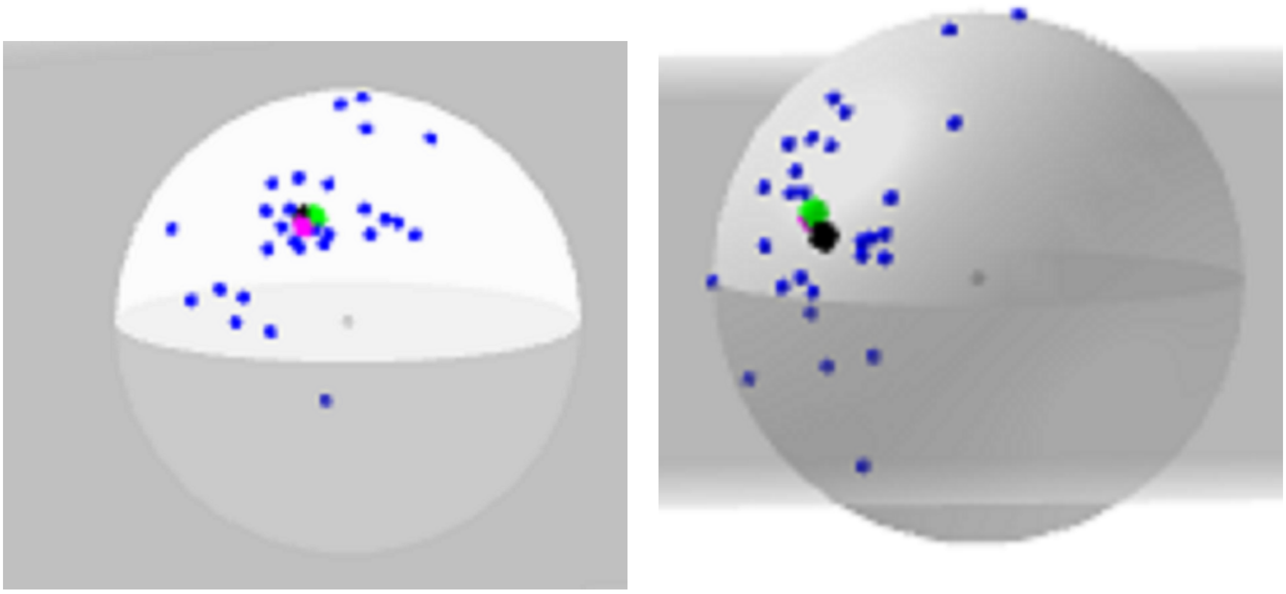

Figure 2: Spherical mean, MTCE and spherical median for Boys' Data for (Left): Frank System and (Right:) McFee System.

for comparison of spherical mean with the MTCE. The corresponding values are tabulated in Table 5. It can be seen from the table that in case of Frank System in both the cases, the estimators are very close to each other, while for the McFee case, the estimators are a bit far. Figures 2 and 3 show the figures for the data analysis of vector cardiogram data.

7 Concluding remarks

In contrast, the *circular variance* is a measure of the spread of a set of angles. It is defined as

$$Var_C(\theta) = 1 - \bar{R}, \tag{7.1}$$

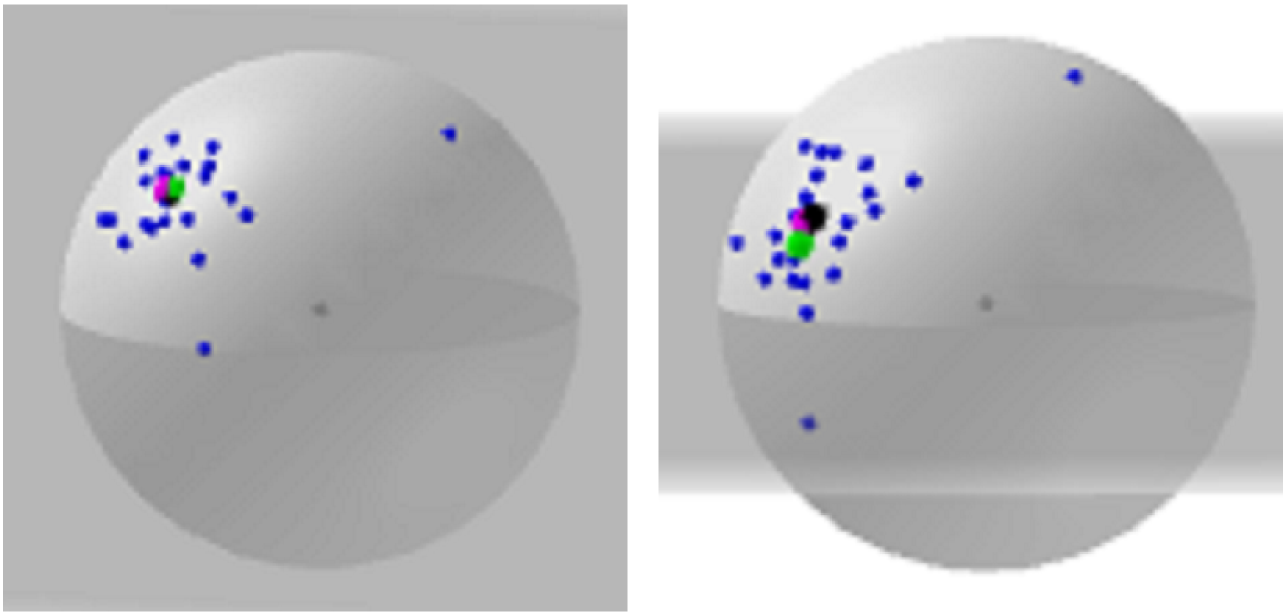

Figure 3: Spherical mean, MTCE and spherical median for Girls' Data for (Left): Frank System and (Right:) McFee System.

where $\bar{R} = R/n$, where n is the number of observations, and R is given by

$$R^2 = \left(\sum_{j=1}^{n} \cos \theta_{x_j} \right)^2 + \left(\sum_{j=1}^{n} \sin \theta_{x_j} \right)^2 . \tag{7.2}$$

Clearly, the *circular variance* ranges between 0 to 1, with the lower the value indicating tighter clustering of the values about a single mean value. However, it is easy to check that the *circular variance* cannot be changed arbitrarily by simply arbitrarily choosing one or two out of n observations. The CBDP of *circular variance* is $(\left[\frac{n}{2}\right] + 1)/n$.

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