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Abstract

A machine or production system is subject to random failure and it is replaced by a new one, and the process repeats. However, replacing the unit at failure may produce hazardous effects, and hence there is a need to replace it according to some replacement policy before the failure occurs (preventive replacement). Also the cost associated with each replacement due to failure is much higher than preventive replacement. Thus, there is an incentive for a controller to attempt to replace before failure occurs. In this paper, we consider the problem of finding an optimal control strategy that balances the cost of replacement with the cost of failure and results in a minimum expected cost per unit time under cumulative damage model with strength degradation. Theoretical evaluation of this expected cost per unit time is complicated and hence evaluation of optimal replacement strategy is suggested using simulation method.

Keywords: Expected cost rate, Simulation, Shock arrival process, Grid search, Simulated annealing

1 Introduction

The units or systems such as machines used in construction, chemical plants, power plants, heavy electrical and mechanical engineering, parts of vehicles, etc., are often subject to shocks in the course of their operation. These shocks may be assumed to appear at random points in time according to a point process and each shock causes some random amount of damage to the operating unit. The unit or system may fail at some sudden shock or it may withstand the shocks until the total damage caused by the shocks exceeds a critical level. The latter one is often encountered in practical situations and can be studied using a cumulative damage model. In this model, the damage caused in the form of crack growth, creep, fatigue, wear, etc., is accumulated until it becomes greater than a pre-specified failure level. Some real life scenarios where this model turns out to be very helpful are discussed below.

Crack in a vehicle axle caused by overload, jerk, etc., grows as long as it is above a certain depth and the axle breaks after that. Scarf et al. (1996) used a stochastic model under periodic inspection to study crack growth. Stochastic models were applied to study fatigue damage of materials by Sobczyk (1987) and Sobczyk and Spencer (1992). In storage batteries, the electric power stored by chemical change is drawn out according to need. Besides, the battery capacity degrades over time due to the continuous oxidation and deoxidation going on inside it. This is treated as an additional damage that continuously increases with time along with the damage due to shocks. The devices suffering from these damages can be modeled using a cumulative damage model with time deterioration. Replacement problem for such models was investigated by Satow et al. (2000). Similarly, as a result of frequent updation of a database system, un-accessed data accumulates as garbage and the system collapsed as soon as it exceeds the tolerance level (Endharta and Yun, 2014; Nakagawa, 2007, p-131).

There has been ample research on the optimum replacement strategy assuming cumulative damage model with a fixed strength or failure level (Nakagawa, 2007, ch-3). In practice, an operating unit is affected by human errors, material quality and operating conditions, etc., and the unit's capacity to withstand damage due to shocks may decrease as its operating time increases. Hence, the strength of a unit may reasonably be described by a deterministic curve which is decreasing in time. Recently, computation and estimation of reliability under such cumulative damage model has been considered by Bhuyan and Dewanji (2017b, 2017a). Keeping the unit or system functional until its failure may turn out to be cost-ineffective and lead to hazardous situations. If the axle of an automobile breaks in the course of its journey, then it may cost in terms of human lives, the goods it carries and extra money to repair. It creates a havoc among the users when servers in large systems such as banks, railways, online application programmes, etc., become unresponsive which often happens due to garbage created inside the database. Failure of units in nuclear power plants has proven its fatality in some events in the recent past. Hence, there is a need for preventive maintenance of the units before failure occurs (Nakagawa, 2005).

There has been considerable amount of research on the replacement policies for cumulative damage models. Taylor (975), Zuckerman (1977), and Chikte and Deshmukh (1981) discussed optimal replacement policies under additive damage model. Nakagawa and Kijima (1989) considered the replacement problem for cumulative damage model with minimal repair at failure. A damage based replacement policy has been addressed by Feldman (1976) and Nakagawa (1976). Satow and Nakagawa (1997) investigated the replacement policies for a cumulative damage model where the unit suffers from two kinds of damage. A good summary of the basic and modified replacement policies along with garbage collection and data backup policies for a database system can be found in Nakagawa (1976). Recently, Endharta and Yun (2014) provided a numerical comparison of the basic replacement policies under additive damage model. In all these replacement problems considered so far, the cumulative damage model has been assumed to have a fixed strength. We, in this article, have discussed the replacement policies for the cumulative damage model having strength that is continuously non-increasing over time. Even if the distribution functions of both inter-arrival time between successive shocks and damage due to each shock possess closure property under convolution, expression for the expected cost rate (i.e., expected cost per unit time) involves integrals and infinite sums, numerical evaluation of which is difficult. Complexity of computation increases if closed form expressions for the convolution of the associated distribution functions are not available and the strength is time dependent (See Section 3). In order to avoid such difficulty, Nakagawa (1976) and Endharta and Yun (2014) assumed constant strength and independent and identically distributed (iid) Exponential distributions for the successive inter-arrival times and damages so that the related convolutions follow the respective Gamma distributions. In this article, we propose a simulation based method for evaluation of the optimal replacement strategy which provides flexibility in choosing the distribution functions for both inter-arrival time between successive shocks and damage due to each shock. Therefore, the domain of application of the proposed method is much wider.

The rest of the paper is organized as follows. In the next section, we discuss the preliminaries which include the notation and assumptions regarding the proposed modeling framework. In Section 3, we present the mathematical formulations for the basic replacement policies with different optimization criteria. Section 4 deals with the different computational methods and the issues therein. Some numerical results for different choices of the damage and inter-arrival time distributions, strength of the unit, etc., are presented in Section 5. In Section 6, we consider some generalizations of the damage distribution and the numerical results in those cases. Finally, we conclude the paper with some remarks in Section 7.

2 Preliminaries

We assume that the operating unit starts working at time 0 and its initial damage level is 0. As it goes on working, it is subject to shocks and suffers from some amount of damage due to each shock. These damages caused by the successive shocks are accumulated over time. Let $N(t)$ represent the number of shocks by time t. It is assumed that the shocks arrive according to a renewal process. Let $X_1, X_2, \ldots$ be the sequence of independent and identically distributed random variables which denote the inter-arrival times between successive shocks having the common distribution function $F(\cdot)$. Then $S_j = \sum_{i=1}^{j} X_i,\ j \geq 1$ represents the arrival time of the jth shock and has the distribution function $F^{(j)}(\cdot)$, where $F^{(j)}(\cdot)$ is the j-fold convolution of $F(\cdot)$ with itself. The damage due to the jth shock is denoted by W_j. These W_j $(j = 1, 2, \ldots)$ are assumed to be independent and identically distributed and also independent of the shock arrival process $N(t)$ (that is, the X_i's). Let $W_j,\ j \geq 1$, have a common distribution function $G(\cdot)$. Then the total damage at the jth shock will have the distribution function $G^{(j)}(\cdot)$, the j-fold convolution of $G(\cdot)$ with itself.

The strength of the unit is described by $K(t)$ which is continuous and decreasing in time t. Note that, under the present stress-strength interface, there are two different types of failure modes, either due to strength degradation at or below the existing level of accumulated stress, or due to arrival of a shock resulting in the increased stress exceeding or equaling the strength at that time. Then a unit fails when its strength reduces to zero even if no shock arrives by that time. One needs to consider corrective replacement of the unit with a new one immediately after failure. According to the existing basic replacement policies, the unit is preventively replaced before failure at a planned time T, shock number N, or a damage level Z, whichever occurs first; otherwise it is replaced at failure. In our work, we have adopted the basic replacement policies with an additional condition $Z \leq K(T)$ so that the damage level Z has some relevance in deciding the replacement policies. If the total damage at the Nth shock exceeds the pre-specified damage level Z, or the strength at that time of shock arrival, then it is assumed that the replacement of the unit is due to damage, or failure, as is the case, instead of the shock number N. This assumption is reasonable if both the replacement costs, due to damage Z and due to failure, are greater than that due to shock number N, in order to safeguard the worse situation. Similarly, if the total damage at the Nth shock exceeds both the damage level Z and the strength at that time of shock arrival, we assume that the replacement is due to the failure, since that is the most expensive of the three.

Let us denote the probabilities that the unit is replaced at scheduled time T, shock number N, damage level Z and at failure, by p_T, p_N, p_Z and p_K, respectively. There is cost associated with each replacement with the cost of corrective replacement being higher than those of the preventive replacement. If c_T, c_N, c_Z, c_K are the costs incurred from replacement at time T, shock number N, damage level Z and at failure, respectively, then c_K is higher than each of c_T, c_N and c_Z with $c_Z > c_N$. The expected cost for replacement can be obtained as a function of T, N and Z, say $C(T,\tilde{N},Z)$, which upon division by the mean time to replacement gives the expected replacement cost per unit time, termed as the 'expected cost rate' for brevity. The expected cost rate is a reasonable objective function to minimize for finding the optimal policies for replacement.

3 Replacement Policies

As described in the previous section, a preventive replacement is to be carried out at a planned time T, or at a shock number N, or at a damage level Z, whichever occurs first. As in Satow et al. (2000), we first consider these three design variables T, N and Z one at a time and consider the corresponding expected cost rates as the objective function to minimize. Thereafter, we consider all these three variables simultaneously. For this purpose, we derive the expected cost rates for replacement separately as a function of T, N and Z and then all taken together.

3.1 Replacement at T

The preventive replacement of the unit is done only at a planned time T. The unit is replaced either at T or at failure, whichever occurs first. There is no replacement at the Nth shock or the cumulative damage reaching Z. As discussed in the previous section, we assume that the replacement is corrective rather than preventive, if failure happens at time T. The probability of preventive replacement p_T due to reaching age T prior to failure occurrence can be obtained as

$$\begin{aligned} p_T &= \sum_{j=0}^{\infty} P\left[S_j \leq T, S_{j+1} > T, W_1 + \cdots + W_j \leq K(T)\right] \\ &= \sum_{j=0}^{\infty} \left[F^{(j)}(T) - F^{(j+1)}(T)\right] G^{(j)}(K(T)). \end{aligned}$$

Since the unit is replaced either at the planned time T or at failure, therefore the probability that the unit is replaced at failure is given by $p_K = 1 - p_T$. If c_T and c_K are the costs incurred when the unit is replaced at T and at failure, respectively, then the expected cost of replacement can be written as

$$\tilde{C}_1(T) = c_K - (c_K - c_T) \sum_{j=0}^{\infty} \left[F^{(j)}(T) - F^{(j+1)}(T)\right] G^{(j)}(K(T)). \tag{1}$$

If S denotes the time to replacement, then for any $t \in (0, T]$

$$\begin{aligned} P\left[S > t\right] &= \sum_{j=0}^{\infty} P\left[S_j \leq t, S_{j+1} > t, W_1 + \cdots + W_j \leq K(t)\right] \\ &= \sum_{j=0}^{\infty} \left[F^{(j)}(t) - F^{(j+1)}(t)\right] G^{(j)}(K(t)). \end{aligned}$$

Thus,

$$P\left[S > t\right] = \begin{cases} \sum_{j=0}^{\infty} \left[F^{(j)}(t) - F^{(j+1)}(t)\right] G^{(j)}(K(t)), & t \leq T \\ 0, & t > T. \end{cases}$$

The mean time to replacement for this case will be given by

$$\int_0^{\infty} P\left[S > t\right] dt = \sum_{j=0}^{\infty} \int_0^T \left[F^{(j)}(t) - F^{(j+1)}(t)\right] G^{(j)}(K(t)) dt. \tag{2}$$

Thus the expected cost rate $C_1(T)$, when the unit is replaced either at T or at failure, can be obtained, dividing (1) by (2), as

$$C_1(T) = \frac{c_K - (c_K - c_T) \sum_{j=0}^{\infty} \left[F^{(j)}(T) - F^{(j+1)}(T)\right] G^{(j)}(K(T))}{\sum_{j=0}^{\infty} \int_0^T \left[F^{(j)}(t) - F^{(j+1)}(t)\right] G^{(j)}(K(t)) dt}.$$

3.2 Replacement at N

The operating unit is replaced either at the planned shock number N or at failure, whichever occurs first. There is no replacement due to a planned time T or a damage level Z. As discussed before, we assume that the replacement is corrective rather than preventive, if failure happens at the arrival of the Nth shock. The probability that the unit is replaced at the Nth shock prior to failure occurrence is

$$\begin{aligned} p_N &= P\left[\sum_{j=0}^{N} W_j \le K(S_N)\right] \\ &= \int_0^\infty P\left[\sum_{j=0}^{N} W_j \le K(s)\right] dP\left[S_N \le s\right] \\ &= \int_0^\infty G^{(N)}(K(s))dF^{(N)}(s). \end{aligned}$$

Similar to the previous case, the probability of replacement at failure is given by $p_K = 1 - p_N$. The costs of replacement at the Nth shock and at failure are assumed to be c_N and c_K, respectively. Then the expected cost $\tilde{C}_2(N)$ of replacement can be written as

$$\tilde{C}_2(N) = c_K - (c_K - c_N)\int_0^\infty G^{(N)}(K(s))dF^{(N)}(s). \tag{3}$$

For any $t \in [0, \infty)$, the probability that the unit is not replaced before time t is given by,

$$\begin{aligned} P\left[S > t\right] &= \sum_{j=0}^{N-1} P\left[S_j \le t, S_{j+1} > t, W_1 + \cdots + W_j \le K(S_j)\right] \\ &= \sum_{j=0}^{N-1} \int_0^t G^{(j)}(K(s))\bar{F}(t-s)dF^{(j)}(s). \end{aligned}$$

Then the mean time to replacement in this case will be

$$\int_0^\infty P\left[S > t\right] dt = \sum_{j=0}^{N-1} \int_0^\infty \int_0^t G^{(j)}(K(s))\bar{F}(t-s)dF^{(j)}(s)dt. \tag{4}$$

Thus, dividing the expected cost in (3) by the mean time to replacement in (4), the expected cost rate $C_2(N)$ for replacement is given by

$$C_2(N) = \frac{c_K - (c_K - c_N)\int_0^\infty G^{(N)}(K(s))dF^{(N)}(s)}{\sum_{j=0}^{N-1}\int_0^\infty \int_0^t G^{(j)}(K(s))\bar{F}(t-s)dF^{(j)}(s)dt}.$$

3.3 Replacement at Z

The problem of replacement at Z needs to be looked at in a bit different way from those for replacement at time T or at shock number N. Let T_0 be the time such that $K(T_0) = Z$. Thus, before T_0, the replacement of the unit can either be due to the damage level Z or due to failure; but after T_0 the replacement will be only due to failure of the unit. As discussed before, we assume that the replacement is corrective rather than preventive, if the accumulated damage exceeds both Z and the strength at the time of a shock arrival. The probability p_Z that the replacement is done due to damage Z prior to failure occurrence is obtained as

$$\begin{aligned}
p_Z &= \sum_{j=0}^{\infty} P\left[W_1 + \cdots + W_j \le Z < W_1 + \cdots + W_{j+1} \le K(S_{j+1})\right] P\left[S_{j+1} \le T_0\right] \\
&= \sum_{j=0}^{\infty} \int_0^{T_0} P\left[W_1 + \cdots + W_j \le Z < W_1 + \cdots + W_{j+1} \le K(s)\right] dP\left[S_{j+1} \le s\right] \\
&= \sum_{j=0}^{\infty} \int_0^{T_0} \int_0^{Z} P\left[Z - x < W_{j+1} \le K(s) - x\right] dP\left[W_1 + \cdots + W_j \le x\right] dP\left[S_{j+1} \le s\right] \\
&= \sum_{j=0}^{\infty} \int_0^{T_0} \int_0^{Z} \left[G(K(s)-x) - G(Z-x)\right] dG^{(j)}(x) dF^{(j+1)}(s).
\end{aligned}$$

The replacement is done either at damage level Z or at failure. Therefore, as before, the expected cost of replacement can be written as

$$\tilde{C}_3(Z) = c_K - (c_K - c_Z)\sum_{j=0}^{\infty} \int_0^{T_0} \int_0^{Z} \left[G(K(s)-x) - G(Z-x)\right] dG^{(j)}(x) dF^{(j+1)}(s), \tag{5}$$

where c_K and c_Z are the costs incurred from replacement at Z and at failure, respectively. In order to calculate the mean time to replacement, we proceed by first calculating the probability that the unit is not replaced before some time t. To serve our purpose, we need to define a modified replacement level $\tilde{K}(t)$ as given by

$$\tilde{K}(t) = \begin{cases} Z, & t \le T_0 \\ K(t), & t > T_0. \end{cases}$$

Then the probability that replacement is not done during $[0, t]$ will be

$$\begin{aligned} P\left[S > t\right] &= \sum_{j=0}^{\infty} P\left[S_j \le t, S_{j+1} > t, W_1 + \cdots + W_j \le \tilde{K}(S_j)\right] \\ &= \sum_{j=0}^{\infty} \int_0^t P\left[s + X_{j+1} > t, W_1 + \cdots + W_j \le \tilde{K}(s)\right] dP\left[S_j \le s\right] \\ &= \sum_{j=0}^{\infty} \int_0^t G^{(j)}(\tilde{K}(s))\bar{F}(t-s)dF^{(j)}(s). \end{aligned}$$

Therefore, the mean time to replacement is given by

$$\int_0^{\infty} P\left[S > t\right] dt = \sum_{j=0}^{\infty} \int_0^{\infty} \int_0^t G^{(j)}(\tilde{K}(s))\bar{F}(t-s)dF^{(j)}(s)dt. \tag{6}$$

Thus, the expected cost rate for replacement in this case is obtained, dividing (5) by (6), as

$$C_3(Z) = \frac{c_K - (c_K - c_Z)\sum_{j=0}^{\infty}\int_0^{T_0}\int_0^Z \left[G(K(s)-x) - G(Z-x)\right] dG^{(j)}(x)dF^{(j+1)}(s)}{\sum_{j=0}^{\infty}\int_0^{\infty}\int_0^t G^{(j)}(\tilde{K}(s))\bar{F}(t-s)dF^{(j)}(s)dt}.$$

3.4 Replacement at T, N and Z

Preventive replacement takes place at a planned time T, shock number N or at a damage level Z, whichever occurs first. As discussed before, if the cumulative damage at the Nth shock exceeds Z as well as the strength at that time, we assume that the replacement is corrective, since that is more expensive compared to preventive replacement. Also, it is reasonable to restrict the design space of (T, N, Z) into those choices of T and Z such that $Z \le K(T)$, or $T \le T_0$, so that the replacement due to Z remains a possibility. As before, let us write p_T, p_N, p_Z and p_K as the probabilities that the unit is replaced at scheduled time T, shock number N, damage level Z and at failure, respectively. Then

$$p_T = \sum_{j=0}^{N-1} G^{(j)}(Z)\left[F^{(j)}(T) - F^{(j+1)}(T)\right]$$

and

$$p_N = F^{(N)}(T)G^{(N)}(Z).$$

Note that the above expressions of p_T and p_N are exactly same as those obtained in the case of cumulative damage model with fixed strength (Nakagawa, 2007, ch-3). The probability that the unit is replaced at damage level Z can be calculated as

$$\begin{aligned}
p_Z &= \sum_{j=0}^{N-1} P\left[W_1 + \cdots + W_j \le Z < W_1 + \cdots + W_{j+1} \le K(S_{j+1})\right] P\left[S_{j+1} \le T\right] \\
&= \sum_{j=0}^{N-1} \int_0^T P\left[W_1 + \cdots + W_j \le Z < W_1 + \cdots + W_{j+1} \le K(s)\right] dP\left[S_{j+1} \le s\right] \\
&= \sum_{j=0}^{N-1} \int_0^T \int_0^Z P\left[Z - x < W_{j+1} \le K(s) - x\right] dP\left[W_1 + \cdots + W_j \le x\right] dP\left[S_{j+1} \le s\right] \\
&= \sum_{j=0}^{N-1} \int_0^T \int_0^Z \left[G(K(s) - x) - G(Z - x)\right] dG^{(j)}(x) dF^{(j+1)}(s).
\end{aligned}$$

Similarly, the probability that the unit is replaced at failure is

$$\begin{aligned}
p_K &= \sum_{j=0}^{N-1} P\left[W_1 + \cdots + W_j \le Z, W_1 + \cdots + W_{j+1} > K(S_{j+1})\right] P\left[S_{j+1} \le T\right] \\
&= \sum_{j=0}^{N-1} \int_0^T P\left[W_1 + \cdots + W_j \le Z, W_1 + \cdots + W_{j+1} > K(s)\right] dP\left[S_{j+1} \le s\right] \\
&= \sum_{j=0}^{N-1} \int_0^T \int_0^Z P\left[W_{j+1} > K(s) - x\right] dP\left[W_1 + \cdots + W_j \le x\right] dP\left[S_{j+1} \le s\right] \\
&= \sum_{j=0}^{N-1} \int_0^T \int_0^Z \bar{G}(K(s) - x) dG^{(j)}(x) dF^{(j+1)}(s).
\end{aligned}$$

It can be easily verified that $p_T + p_N + p_Z + p_K = 1$. Again, write c_T, c_N, c_Z and c_K as the costs of replacement at the planned time T, shock number N, damage level Z and at failure, respectively, with c_K being the largest. Then the expected cost of replacement of the unit is given by

$$\begin{aligned}\tilde{C}(T,N,Z) &= c_K - (c_K - c_T)\sum_{j=0}^{N-1} G^{(j)}(Z)\left[F^{(j)}(T) - F^{(j+1)}(T)\right] - (c_K - c_N)\,F^{(N)}(T)G^{(N)}(Z) \\ &\quad - (c_K - c_Z)\sum_{j=0}^{N-1}\int_0^T\int_0^Z \left[G(K(s)-x) - G(Z-x)\right] dG^{(j)}(x)dF^{(j+1)}(s). \qquad (7)\end{aligned}$$

For any $t \in (0,T]$, $P\{S > t\}$ is same as the probability that at most $N-1$ shocks occur during $[0,t)$ and the total damage due to those shocks does not exceed the damage level Z. Hence, for any $t \in [0,T)$,

$$\begin{aligned}P[S>t] &= \sum_{j=0}^{N-1} P\left[W_1 + \cdots + W_j \le Z\right] P\left[N(t) = j\right] \\ &= \sum_{j=0}^{N-1} G^{(j)}(Z)\left[F^{(j)}(t) - F^{(j+1)}(t)\right].\end{aligned}$$

Since the operating unit is anyway going to be replaced after the planned time T, the survival function of S can be written as

$$P[S>t] = \begin{cases}\sum_{j=0}^{N-1} G^{(j)}(Z)\left[F^{(j)}(t) - F^{(j+1)}(t)\right], & t < T \\ 0, & t \ge T.\end{cases}$$

Thus the mean time to replacement is given by

$$\int_0^\infty P[S>t]\,dt = \sum_{j=0}^{N-1} G^{(j)}(Z)\int_0^T \left[F^{(j)}(t) - F^{(j+1)}(t)\right] dt. \qquad (8)$$

The expected cost rate of replacement denoted by $C(T,N,Z)$ can be obtained, dividing (7) by (8), as

$$
\begin{aligned}
C(T,N,Z) \;=\; & \Bigg[c_K - (c_K - c_T)\sum_{j=0}^{N-1} G^{(j)}(Z)\left[F^{(j)}(T) - F^{(j+1)}(T)\right] \\
& - (c_K - c_Z)\sum_{j=0}^{N-1}\int_0^T\int_0^Z \left[G(K(s)-x) - G(Z-x)\right] dG^{(j)}(x) dF^{(j+1)}(s) \\
& - (c_K - c_N)\, F^{(N)}(T) G^{(N)}(Z)\Bigg] \Big/ \left[\sum_{j=0}^{N-1} G^{(j)}(Z)\int_0^T \left[F^{(j)}(t) - F^{(j+1)}(t)\right] dt\right]. \quad (9)
\end{aligned}
$$

Our objective is to find the optimum choice of $\hat{T}$, $\hat{N}$ and $\hat{Z}$ which minimizes the expected cost rate $C(T,N,Z)$ in the restricted design space. Theoretically optimizing the expected cost rates leads to complicated expressions and requires imposing more conditions which are practically less important (Nakagawa, 2007, ch-3). Thus, there is a need to go for numerical investigation for finding the optimum replacement policy. The methods and the issues associated with this investigation are discussed in the following section.

4 Computational Issues

As mentioned in the previous section, the expressions for expected cost rates are difficult to be analytically dealt with. One of the major problems in computing the expressions is evaluation of the convolution of the distribution functions. The simplest of the cases is when both of the inter-arrival time between successive shocks and the damage due to each shock follow exponential distribution as they possess closure property under convolution. If the distribution functions do not have closure property under convolution, then inversion method serves well in computing these convolutions.

4.1 Gil-Pelaez Inversion Formula

The characteristic function of a random variable X having distribution function $G(\cdot)$ is given by $\phi(u) = \int_{-\infty}^{\infty} e^{iux} dG(x)$. The distribution function $G(x)$ can be obtained from the characteristic function $\phi(u)$ using a version of the inversion formula given in Gil-Pelaez (1959). For a continuity point x of $G(x)$,

$$\begin{aligned} G(x) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \left(\frac{e^{-iux}\phi(u) - e^{iux}\phi(-u)}{2iu}\right) du \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Im\left(\frac{e^{-iux}\phi(u)}{u}\right) du, \end{aligned} \tag{1}$$

where $Im(z)$ denotes the imaginary part of a complex number z. The characteristic function of $Y = \sum_{i=1}^n X_i$, where $X_1, \ldots, X_n$ are independent and identically distributed with a common distribution function $F(\cdot)$ and characteristic function $\phi_X(\cdot)$, is given by $\phi_Y(u) = \{\phi_X(u)\}^n$. The above result is useful in evaluating the distribution function of Y using numerical integration (Witkovsky, 2001). Then, using (1), the distribution function $F^{(n)}(\cdot)$ can be written as

$$F^{(n)}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Im\left(\frac{e^{-iux}\phi_Y(u)}{u}\right) du \tag{2}$$

Inversion of the characteristic function can be done using numerical integration. Any standard software package equipped with numerical integration (e.g.- *integrate* in the package R) can be used in this direction. The details of the algorithm and its precision are discussed in Piessens et al. (1983).

Even if we succeed in computing the convolutions, evaluation of the expected cost rates itself is another big challenge. The expressions for expected cost rates involves infinite sums or infinite integrals, or both in some cases. Evaluation of the integrals can be carried out using numerical integration as discussed previously. On the other hand, the infinite sums can be approximated by taking large number, say 10000, of terms and ignoring the terms after that. Evaluation of the expected cost rate using this approach is computationally challenging even if both of the inter-arrival time between successive shocks and damage due to each shock follow exponential distributions. Complexity of computation increases if there is no closed form for the characteristic function of damage distribution (e.g., Weibull, Log-normal, etc.). Apart from that, the behaviour of the integrand in the inversion formula is extremely fluctuating for certain choices of distribution and associated parameters.

4.2 Simulation Method

In this method, the whole process of shock arrivals and accumulation of damages as against the degradation of strength is virtually created. For a fixed T, N and Z, the proposed algorithm gives as output one realization each for the time to replacement T_R and a variable I_R indicating whether the replacement is due to failure or due to one of T, N and Z. The mean time to replacement and the probabilities of replacement can be calculated by simulating a large number, say 10000, of realizations of T_R and I_R. The algorithm for simulating a realization for each of T_R and I_R is given below :

$X_0 = 0$, $W_0 = 0$; For $i = 1, 2, \ldots,$

Step 1. Simulate $X_i \sim F(\cdot)$ and $W_i \sim G(\cdot)$;

Step 2. Calculate $S_i = \sum_{j=0}^{i} X_j$ and $L_i = \sum_{j=0}^{i} W_j$;

Step 3. If $L_i < \tilde{K}(S_i)$, then next i (i.e. repeat Step 1 and Step 2);
else, if $L_{i-1} \leq \tilde{K}(S_i) < L_i$, then set $\tilde{T} = S_i$;
else, find $\tilde{T} = t$ by solving $\tilde{K}(t) = L_{i-1}$;

Step 4. If $N(\tilde{T}) > N$, then $T_R = \min\{T, S_N\}$ and $I_R = 1\mathbb{1}(T_R = S_N) + 2\mathbb{1}(T_R = T)$;
else, if $\tilde{T} < T_0$ and $L_i < K(S_i)$, then $T_R = \min\left\{\tilde{T}, T\right\}$ and $I_R = 2\mathbb{1}(T_R = T) + 3\mathbb{1}(T_R = \tilde{T})$;
else, if $\tilde{T} < T$, then $T_R = \tilde{T}$ and $I_R = 0$;
else, $T_R = T$ and $I_R = 2$.

Now, the expected cost rate $C(T, N, Z)$ given in (9) is approximated as $\frac{c_T \bar{I}_T + c_Z \bar{I}_Z + c_N \bar{I}_N + c_K(1 - \bar{I}_T - \bar{I}_Z - \bar{I}_N)}{\bar{T}_R}$, where $\bar{I}_T$, $\bar{I}_Z$, $\bar{I}_N$ and $\bar{T}_R$ are mean of $I_T = \mathbb{1}(I_R = 2)$, $I_Z = \mathbb{1}(I_R = 3)$, $I_N = \mathbb{1}(I_R = 1)$ and T_R, respectively, based on 10000 simulated observations. The proposed algorithm is repeated for different choices of T, N and Z for evaluation of the expected cost rate as a function of T, N and Z which can then be minimized for finding the optimal values of T, N and Z for replacement. The optimal replacement, while considering one of T, N and Z (See Subsections 3.1-3.3), can be determined by assuming the other two to be infinite in the simulation method. The minimum expected cost rate can be obtained by using the methods of grid search, simulated annealing, etc. The method of simulated annealing has been implemented to escape a local minimum with certain probability in order to search for the global minimum. Interested readers can see S. Kirkpatrick and Vecchi (1983) and Dowsland (1995) for more detailed study on simulated annealing. The programming and associated computation for simulation method is much simpler than those required for implementation of the inversion method. More importantly, the domain of application is much wider providing flexibility in choosing both the distribution functions for inter-arrival time between successive shocks and damage due to each shock.

5 Numerical Results

The computations have been done under different distributional assumptions with several sets of values for the associated parameters, different strength degradations and the cost incurred from replacement at failure. In all of the computations, the costs incurred from preventive replacements at T, N or Z are assumed to be 1, i.e. $c_T = c_N = c_Z = 1$. The inter-arrival time between successive shocks has been assumed to follow (*i*) Exponential distribution with mean $1/\lambda$, denoted by $Exp(\lambda)$, and (*ii*) Log-normal distribution with Normal parameters μ and σ, denoted by $LN(\mu, \sigma)$, with mean being $\exp\left(\mu + \frac{1}{2}\sigma^2\right)$. The distribution functions for the damage caused by each shock has been assumed to be either (*i*) Exponential with mean $1/\mu$, denoted by $Exp(\mu)$, or (*ii*) Weibull with scale parameter α and shape parameter β, denoted by $Wei(\alpha, \beta)$, with mean damage being $\alpha\Gamma\left(1 + \frac{1}{\beta}\right)$. The cumulative damage model is considered to have strength $K(t)$ that degrades exponentially, linearly or remains constant over time. In Table 1, we present the optimum values $\hat{T}$, $\hat{N}$ and $\hat{Z}$ which minimize the expected cost rates $C_1(T)$, $C_2(N)$ and $C_3(Z)$, respectively, along with the corresponding minimum expected cost rates. Then, in Table 2 and Table 3, we present the optimum values of $\hat{T}$, $\hat{N}$ and $\hat{Z}$ by minimizing the expected cost rate $C(T, N, Z)$ as a function of T, N and Z, along with the corresponding minimum expected cost rate. The results are obtained by implementing both the grid search and the simulated annealing algorithm. As expected, one can observe that the optimal values $\hat{T}$, $\hat{N}$ and $\hat{Z}$ decrease as cost of corrective replacement c_K increases (See Table 1). Also, as expected, the optimal values of T, N and Z in Table 2 are larger compared to those in Table 1. Interestingly, the minimum cost rate is smaller for the simultaneous optimization of T, N and Z compared to those of individual cases.

Table 1: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and corresponding minimal cost rates $C_1(\hat{T})$, $C_2(\hat{N})$ and $C_3(\hat{Z})$ with $c_T = c_N = c_Z = 1$. Means of the relevant distributions given in parentheses.

$K(t)$	F	G	c_K	$\hat{T}$	$C_1(\hat{T})$	$\hat{N}$	$C_2(\hat{N})$	$\hat{Z}$	$C_3(\hat{Z})$
$100\exp(-0.1t)$	$Exp(0.4)$	$Exp(4)$	2	29.42	0.036	10	0.044	2.45	0.046
	(2.5)	(0.25)	4	28.23	0.037	9	0.049	1.84	0.056
			6	27.29	0.037	9	0.053	1.70	0.060
$\max\{50-t,0\}$	$Exp(0.5)$	$Exp(0.5)$	2	20.53	0.057	10	0.058	16.15	0.065
	(2)	(2)	4	17.19	0.067	9	0.066	15.07	0.066
			6	16.03	0.071	8	0.070	14.39	0.070
10	$Exp(0.5)$	$Exp(1)$	2	20.21	0.084	9	0.078	7.95	0.063
	(2)	(1)	4	12.76	0.119	6	0.100	6.92	0.071
			6	10.83	0.139	6	0.113	6.57	0.078
$150\exp(-0.05t)$	$LN(2,1)$	$Wei(10,15)$	2	26.09	0.042	3	0.046	21.13	0.046
	(12.18)	(9.66)	4	21.96	0.047	2	0.062	13.16	0.062
			6	21.85	0.049	2	0.074	13.90	0.074
$\max\{60-t,0\}$	$LN(1,1)$	$Wei(10,5)$	2	15.47	0.089	4	0.073	30.25	0.072
	(4.48)	(9.18)	4	11.56	0.108	3	0.086	24.74	0.086
			6	9.72	0.120	3	0.095	22.59	0.095
50	$LN(2,1)$	$Wei(10,15)$	2	74.72	0.028	5	0.019	39.63	0.018
	(12.18)	(9.66)	4	35.18	0.038	4	0.021	39.30	0.018
			6	29.84	0.043	4	0.021	37.71	0.018

Table 2: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and minimum expected cost rate $C(\hat{T},\hat{N},\hat{Z})$ with $c_T = c_N = c_Z = 1$. Means of the relevant distributions given in parentheses.

$K(t)$	F	G	c_K	Method	$\hat{T}$	$\hat{N}$	$\hat{Z}$	$C(\hat{T},\hat{N},\hat{Z})$
$100\exp(-0.1t)$	$Exp(0.4)$	$Exp(4)$	4	Grid Search	31.40	19	4.21	0.033
	(2.5)	(0.25)		Simulated Annealing	30.99	18	4.20	0.034
$\max\{50-t,0\}$	$Exp(0.5)$	$Exp(0.5)$	6	Grid Search	25.01	13	20.40	0.051
	(2)	(2)		Simulated Annealing	25.03	13	19.91	0.051
$150\exp(-0.05t)$	$LN(2,1)$	$Wei(10,15)$	2	Grid Search	35.02	4	25.87	0.036
	(12.18)	(9.66)		Simulated Annealing	34.12	4	24.69	0.037
$\max\{60-t,0\}$	$LN(1,1)$	$Wei(10,5)$	4	Grid Search	30.41	4	23.74	0.067
	(4.48)	(9.18)		Simulated Annealing	30.72	4	23.06	0.067

Table 3: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and minimum expected cost rate $C(\hat{T}, \hat{N}, \hat{Z})$ with $c_K = 6, c_T = 0.5, c_N = 1.5, c_Z = 1$. Means of the relevant distributions given in parentheses.

$K(t)$	F	G	Method	$\hat{T}$	$\hat{N}$	$\hat{Z}$	$C(\hat{T}, \hat{N}, \hat{Z})$
$100\exp(-0.1t)$	$Exp(0.4)$	$Exp(4)$	Grid Search	28.57	26	5.41	0.018
	(2.5)	(0.25)	Simulated Annealing	28.89	26	5.38	0.018
$\max\{50 - t, 0\}$	$Exp(0.5)$	$Exp(0.5)$	Grid Search	28.68	13	19.10	0.033
	(2)	(2)	Simulated Annealing	28.95	13	18.49	0.033
$150\exp(-0.05t)$	$LN(2, 1)$	$Wei(10, 15)$	Grid Search	22.72	8	45.10	0.024
	(12.18)	(9.66)	Simulated Annealing	22.05	8	44.29	0.024
$\max\{60 - t, 0\}$	$LN(1, 1)$	$Wei(10, 5)$	Grid Search	13.41	7	37.01	0.055
	(4.48)	(9.18)	Simulated Annealing	13.93	7	37.32	0.055

6 Some Generalizations

In this section, we consider some generalizations in the assumption related to the successive damage distributions which may be more realistic in some situations. The damages due to shocks may be either dependent or independent but not identically distributed. As we move on to these generalized scenarios, the computational difficulty associated with the inversion method also increases. In such situations, the simulation method turns out to be more effective. The algorithm for simulation remains similar to that described in Section 4.2 except for the damage distributions for simulating the W_i's which change accordingly. The optimal values of T, N and Z and the corresponding minimum expected cost rates are evaluated in the same manner.

6.1 Independent but Non-iid Damage Distributions

Here we assume that the damages caused by the successive shocks may be independent but not identically distributed. For instance, there may be situations where the successive shocks cause damages which are stochastically larger than those due to the preceding ones. Note that when the damages $X_1, X_2, \ldots$ are independent but not identically distributed, then $Y = \sum_{i=0}^{n} X_i$ has the characteristic function given by $\phi_Y(u) = \prod_{i=1}^{n} \phi_{X_i}(u)$, where $\phi_{X_i}(\cdot)$ is the characteristic function of X_i. The characteristic function may be helpful in evaluating the convolutions using method of inversion, but the method of simulation seems to be preferred for reasons discussed earlier. The algorithm for the simulation method remains the same except that the successive damages are now generated from the non-identical distributions.

The values of the optimal T, N and Z and the corresponding minimum values of expected cost rates $C_1(T)$, $C_2(N)$ and $C_3(Z)$ under different distributional assumptions are presented in Table 4. The shocks are assumed to arrive according to a renewal process, i.e. the inter-arrival time between successive shocks are iid with a common distribution function $F(\cdot)$. We have chosen the inter-arrival time distribution to be (i) Exponential distribution with mean $1/\lambda$, denoted by $Exp(\lambda)$, (ii) Log-normal distribution with Normal parameters μ and σ, denoted by $LN(\mu, \sigma)$. Unlike the case of iid damages, here it is assumed that the damage due to ith shock has a distribution function $G_i(\cdot)$. The choices for $G_i(\cdot)$ are (i) Gamma with scale parameter θ_i and shape parameter δ, denoted by $Ga(\theta_i, \delta)$, with mean being $\delta\theta_i$ or (ii) Weibull with scale parameter α_i and shape parameter β, denoted by $Wei(\alpha_i, \beta)$. The computations are done for the cases when the strength of the system $K(t)$ is decreasing with time both exponentially and linearly. As before, the values of c_T, c_N and c_Z are kept unchanged, i.e. $c_T = c_N = c_Z = 1$, and different choices for the costs incurred from replacement at failure have been considered.

Table 4: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and corresponding minimal cost rates $C_1(\hat{T})$, $C_2(\hat{N})$ and $C_3(\hat{Z})$ for independent but not identically distributed damages. Means of the relevant distributions given in parentheses.

$K(t)$	F	G_i	c_K	$\hat{T}$	$C_1(\hat{T})$	$\hat{N}$	$C_2(\hat{N})$	$\hat{Z}$	$C_3(\hat{Z})$
$50\exp(-0.05t)$	$Exp(2)$ (0.5)	$Ga(0.5 + ((i-1) \times 0.1), 5)$ $(2.5 + (i-1) \times 0.1)$	4	1.92	0.725	4	0.571	32.66	0.442
$\max\{60-t, 0\}$	$Exp(0.4)$ (2.5)	$Ga(0.5 \times (0.6)^{i-1}, 5)$ $(2.5 \times (0.6)^{i-1})$	4	5.26	0.359	2	0.207	17.37	0.213
$50\exp(-0.05t)$	$LN(2,1)$ (12.18)	$Wei(10 \times (0.6)^{i-1}, 15)$ $(150 \times (0.6)^{i-1})$	2	17.96	0.059	2	0.065	11.7	0.065
$\max\{60-t, 0\}$	$LN(2,1)$ (12.18)	$Wei(10 \times (1.5)^{i-1}, 5)$ $(50 \times (1.5)^{i-1})$	4	15.13	0.083	2	0.069	13.77	0.069

Under similar distributional assumptions, we have calculated the optimum values $\hat{T}$, $\hat{N}$ and $\hat{Z}$

Table 5: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and minimum expected cost rate $C(\hat{T}, \hat{N}, \hat{Z})$ for independent but not identically distributed damages. Means of the relevant distributions given in parentheses.

$K(t)$	F	G_i	c_K	Method	$\hat{T}$	$\hat{N}$	$\hat{Z}$	$C(\hat{T}, \hat{N}, \hat{Z})$
$50\exp(-0.05t)$	$Exp(2)$	$Ga(0.5 + (i-1) \times 0.1, 5)$	4	Grid Search	4.91	7	33.97	0.412
	(0.5)	$(2.5 + (i-1) \times 0.1)$		Simulated Annealing	4.35	7	35.45	0.425
$\max\{60 - t, 0\}$	$Exp(0.4)$	$Ga(0.5 \times (0.6)^{i-1}, 5)$	4	Grid Search	24.02	3	14.83	0.178
	(2.5)	$(2.5 \times (0.6)^{i-1})$		Simulated Annealing	23.92	3	14.57	0.179
$50\exp(-0.05t)$	$LN(2,1)$	$Wei(10 \times (0.6)^{i-1}, 15)$	2	Grid Search	21.98	4	16.75	0.054
	(12.18)	$(150 \times (0.6)^{i-1})$		Simulated Annealing	22.08	4	16.57	0.053
$\max\{60 - t, 0\}$	$LN(2,1)$	$Wei(10 \times (1.5)^{i-1}, 5)$	4	Grid Search	35.20	3	13.97	0.049
	(12.18)	$(50 \times (1.5)^{i-1})$		Simulated Annealing	34.35	3	14.08	0.053

corresponding to the minimum value of the expected cost rate $C(T, N, Z)$. The results are presented in Table 5.

6.2 Dependent Damage Distribution

In order to model dependent damages, a multivariate damage distribution needs to be considered. We consider a model in which the damage W_i due to the ith shock can be expressed as $W_i = Z_0 + Z_i$, where Z_0 is a random variable representing the minimum damage that arrival of a shock can cause to the unit and Z_i is the additional damage caused by the ith shock depending on its severity, etc.. Then the successive damages $W_1, W_2, \ldots$ become dependent because of the common minimum damage Z_0. If the minimum damage Z_0 and the additional damages Z_i's are assumed to be independent $Ga(\theta_i, 1)$ random variables for $i = 0, 1, 2, \ldots$, then the joint distribution of $W_1, \ldots, W_n$, for given n, is known as the Cheriyan and Ramabhadran's multivariate Gamma distribution (S. Kotz, 2000) whose characteristic function is given by

$$\psi_{W_1,\ldots,W_n}(s) = \left(1 - i\sum_{j=0}^{n} s_j\right)^{-\theta_0} \prod_{j=0}^{n} (1 - is_j)^{-\theta_j} \, ,$$

where $s = (s_1, \ldots, s_n)$. The distribution function of the cumulative damage $U = \sum_{i=1}^{n} W_i$ under these assumptions do not fall into any known class of distributions. But the characteristic function of U is given by

$$\phi_U(l) = (1 - inl)^{-\theta_0}(1 - il)^{-\sum_{j=1}^{n}\theta_j}.$$

This can be helpful in evaluating the convolution of the damage distributions using (2). As we have frequently mentioned, there are several other difficulties in evaluating the expected cost rates since the expressions are not in a closed form. By using the simulation method, we can overcome these complications while having a less computational burden. In this dependent modeling, in particular, the generation of successive damages is simple due to the additive form of the W_i's. The objective, similar to the previous cases, is to find the optimal values of T, N and Z, which result in minimum expected cost rates.

In the following illustrations, as before, we consider the shocks to arrive according to a renewal process with inter-arrival time distribution being (i) Exponential distribution with mean $1/\lambda$, denoted by $Exp(\lambda)$, and (ii) Log-normal distribution with Normal parameters μ and σ, denoted by $LN(\mu, \sigma)$. The dependent damages are assumed to follow Cheriyan and Ramabhadran's multivariate Gamma distribution with parameters θ_0 and $\theta_j = \theta$ for all $j = 1, 2, \ldots$, denoted by $MVGa(\theta_0, \theta)$, with mean damage equal to $\theta_0 + \theta$. The strength of the operating unit can be either exponentially or linearly degrading and the assumptions on the costs incurred from preventive replacement of the unit remains same. The expected cost rate $C(T, N, Z)$ is also minimized as a function of T, N and Z taken simultaneously. The computational burden in the simulation method does not increase much because of the dependent damages. The numerical results are presented in Tables 6 and 7.

7 Concluding Remarks

The cumulative damage model with strength degradation unlike that with a fixed strength has a wider range of applications. However, the replacement problem under such model with decreasing strength has not yet been addressed by any researcher. The unit is preventively replaced before

Table 6: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and corresponding minimal cost rates $C_1(\hat{T})$, $C_2(\hat{N})$ and $C_3(\hat{Z})$ for dependent damage distributions. Means of the relevant distributions given in parentheses.

$K(t)$	F	G	c_K	$\hat{T}$	$C_1(\hat{T})$	$\hat{N}$	$C_2(\hat{N})$	$\hat{Z}$	$C_3(\hat{Z})$
$100\exp(-0.1t)$	$Exp(0.2)$ (5)	$MVGa(0.5, 10)$ (10.5)	2	10.79	0.118	2	0.123	18.91	0.122
$\max\{60-t, 0\}$	$Exp(0.2)$ (5)	$MVGa(10, 5)$ (15)	4	9.59	0.157	2	0.112	24.31	0.104
$100\exp(-0.03t)$	$LN(2, 1)$ (12.18)	$MVGa(0.5, 10)$ (10.5)	2	28.98	0.042	3	0.041	24.76	0.041
$\max\{50-t, 0\}$	$LN(2, 1)$ (12.18)	$MVGa(0.5, 5)$ (5.5)	2	27.61	0.043	3	0.049	12.91	0.05

Table 7: Optimal $\hat{T}$, $\hat{N}$, $\hat{Z}$ and minimum expected cost rate $C(\hat{T}, \hat{N}, \hat{Z})$ for dependent damages. Means of the relevant distributions given in parentheses.

$K(t)$	F	G	c_K	Method	$\hat{T}$	$\hat{N}$	$\hat{Z}$	$C(\hat{T}, \hat{N}, \hat{Z})$
$100\exp(-0.1t)$	$Exp(0.2)$	$MVGa(0.5, 10)$	2	Grid Search	13.62	4	24.88	0.099
	(5)	(10.5)		Simulated Annealing	13.59	4	25.57	0.099
$\max\{60-t, 0\}$	$Exp(0.2)$	$MVGa(10, 5)$	4	Grid Search	25.40	3	22.49	0.088
	(5)	(15)		Simulated Annealing	25.10	3	23.11	0.088
$100\exp(-0.03t)$	$LN(2, 1)$	$MVGa(0.5, 10)$	2	Grid Search	40.19	4	29.71	0.035
	(12.18)	(10.5)		Simulated Annealing	41.61	4	28.46	0.035
$\max\{50-t, 0\}$	$LN(2, 1)$	$MVGa(0.5, 5)$	2	Grid Search	33.89	4	16.10	0.037
	(12.18)	(5.5)		Simulated Annealing	33.94	4	16.01	0.039

failure at a scheduled time T, shock number N and a damage level Z whichever occurs first and correctively replaced at failure. Under this replacement policy, we have obtained the expressions for the expected cost rates of replacement at T, N and Z individually, or all taken together. Those expressions are not in closed form which makes it extremely difficult to analytically derive the optimum conditions. Besides, evaluating the convolutions of the distribution functions itself is a complicated process. The method of inversion can be an aid in computing the convolutions; but we have to take into account the computational complexity that increases while evaluating the expected cost rates. In this work, probably for the first time, the computational issues associated with the replacement problem for cumulative damage model with degrading strength has been discussed. We have proposed a simulation algorithm for evaluating the expected cost rates. The method of simulation reduces the computational burden while providing room for a wider range of distributional choices. We have also considered some generalized cases where the damages caused by shocks can either be dependent or independent but not identically distributed.

In many real life scenarios, initial strength or its path of deterioration over time is random. Sometimes, deterioration of strength over time is due to various environmental causes changing stochastically at every instant. Another possible scenario is that the strength of the operating unit degrades in a non-monotonic fashion. The unit may go through some auto-repairing process that will cause some ups and downs in its strength (Ebrahimi and Ramallingam, 1993). Evaluation of the expected cost rates for replacement in those cases are complicated which adds to the reasons why simulation method should be preferred over other competing methods. Sometimes, continuous observation of operating units may be expensive and time consuming. In such a case, we can go for periodic inspection of the unit where we need to figure out the optimum time interval for inspection.

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