Tampered Random Variable Modeling for Multiple Step-stress Life Test

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Abstract

In this paper, we introduce the Tampered Random Variable (TRV) modeling in multiple step-stress life testing experiments. Here $\tau_1 < \tau_2 < \ldots < \tau_{k-1}$ be $(k-1)$ prespecified time points and $s_1, s_2, \ldots, s_k$ be k prefixed stress levels with s_i being the stress level in force during the time interval $[\tau_{i-1}, \tau_i)$ for $i = 1, \cdots, k$ with $\tau_0 = 0$ and $\tau_k = \infty$. We define the tampered random variable $T_{TRV}^{(k)}$ in multi step-stress scenario and calculate the PDF, CDF, and Hazard rate for the proposed tampered variable $T_{TRV}^{(k)}$. We derive a general expression for the expectation of $T_{TRV}^{(k)}$ under different number k of stress levels and also obtain some results on statistical ordering for different k. All these results are obtained under arbitrary baseline (under normal stress condition with stress level s_1) life distribution. In particular, we consider exponential distribution with mean θ and Weibull distribution with scale parameter λ and shape parameter α for specific expressions. We also prove some results on equivalence of the TRV modeling with the two other existing models for step-stress life testing, namely, Cumulative Exposure (CE) and Tampered Failure rate (TFR). Finally, we consider some variations of the modeling approach for $T_{TRV}^{(k)}$ to suit some specific situations or purposes, including incorporation of the stress levels, discrete life time, bivariate or multivariate life times.

Key words: Life testing experiments, Tampering coefficients, Tampering times, Stress conditions, Baseline lifetime, Stochastic ordering.

1 Introduction

Life testing experiments are in process of development over fifty years or so. Furthermore, in recent years, there has been an increasing interest among manufacturers to produce safe, reliable, and cost effective products. The reliability feature of a product is potentially quite important and is often considered during the design and analysis of the experiments. The main aim of a life testing experiment is to measure one or more reliability characteristics of the product under certain constraint. In such experiments, certain number of identical items are placed on a test under specified operating condition and the failure time of the items are recorded. Due to the substantial improvement of science and technology, most of the present-day items are quite durable. Hence, one of the main difficulties of life testing experiment is that it is very difficult to observe sufficient number of failures in an affordable time under normal operating condition. To overcome such situations, experimenters use alternative techniques. Among them, censoring and accelerated life testing (ALT) play important roles. Censoring is a technique to terminate the experiment in a well-planned manner often before failure of all items takes place. In an ALT, experimental units are put on test under some extreme operational conditions which affect the lifetime of the items adversely so that the items generally fail more quickly than under the normal conditions. The factors which affect the lifetime of an item are called stress factors. For example, voltage, temperature, and humidity could be stress factors for an electronic equipment. In an ALT, the experimenter usually gets to observe more failure within shorter time period thereby cutting down the experimentation cost. A special case of ALT is step stress life testing (SSLT), which provides freedom to changing the stress level in a sequential manner during the experiment.

Let $s_1, s_2, \ldots, s_k$ be k predetermined stress levels, with s_1 usually representing the normal stress condition and $s_2, \cdots, s_k$ usually representing successively accelerated stress conditions, although it need not be so in general. Suppose $0 < \tau_1 < \cdots < \tau_{k-1}$ are the $(k-1)$ prefixed time points when the stress levels are to be accelerated. In a basic form of SSLT, a certain number of units are initially put under test at the normal stress condition with stress level s_1. At time point τ_1, the stress level is changed to s_2 from s_1. Similarly, the stress level is changed to s_3 from s_2 at time point τ_2 and so on. Finally, at the time point

τ_{k-1}, the stress level is changed from s_{k-1} to s_k. A SSLT is called a simple SSLT if $k = 2$ (i.e., there are only two stress levels).

The most popular and commonly used modeling approach for a simple SSLT is known as the Cumulative Exposure (CE) model, first proposed by Sedyakin (1966) and then studied extensively by Nelson (1980). Denoting the two life distributions under the two stress conditions by $F_1(\cdot)$ and $F_2(\cdot)$, respectively, the overall distribution function of a test failure is written as

$$F_{CE}^{(2)}(t) = \begin{cases} F_1(t), & \text{if } t \leq \tau_1, \\ F_2(t - \tau_1 + h_1), & \text{if } t > \tau_1, \end{cases} \tag{1}$$

where $h_1 = F_2^{-1}\{F_1(\tau_1)\}$ which comes from the solution of the equation $F_2(h_1) = F_1(\tau_1)$ so that there is continuity at $t = \tau_1$. Here, the superscript '(2)' in $F_{CE}^{(2)}(t)$ in (1) indicates that the model is for a simple SSLT with $k = 2$. A second modeling approach, known as the tampered failure rate (TFR) model, was first proposed by Bhattacharyya and Soejoeti (1989), which assumes that the effect of changing the stress is to multiply the initial failure rate function $\lambda_1(t)$ by an unknown factor α_1 (usually, greater than 1) after the change point τ_1. Denoting the failure rate function of the overall life length by $\lambda_{TFR}^{(2)}(t)$, the proposed TFR model is defined as

$$\lambda_{TFR}^{(2)}(t) = \begin{cases} \lambda_1(t), & \text{if } t \leq \tau_1, \\ \alpha_1 \lambda_1(t), & \text{if } t > \tau_1, \end{cases} \tag{2}$$

where the factor α_1 (usually, greater than 1) depends on both the stress levels s_1 and s_2 and possibly on τ_1 as well. Goel (1971) first introduced the tampered random variable (TRV) modeling in the context of a simple SSLT (See also DeGroot and Goel (1979), which assumes that the effect of change of the stress level at time τ_1 is equivalent to changing (usually, scaling down) the remaining life of the experimental unit by an unknown positive factor, say β_1 (usually, less than 1). Let T be the random variable representing the life time under normal stress condition with distribution function $F(\cdot)$. Then, the overall lifetime, denoted by the 'tampered' random variable $T_{TRV}^{(2)}$, is defined as

$$T_{TRV}^{(2)} = \begin{cases} T, & \text{if } T \leq \tau_1, \\ \tau_1 + \beta_1(T - \tau_1), & \text{if } T > \tau_1, \end{cases} \tag{3}$$

where the scale factor β_1, called the tampering coefficient, depends on both s_1 and s_2 and possibly on τ_1 as well. The time point τ_1 is called the tampering time.

There have been natural generalizations of the cumulative exposure (CE) and tampered failure rate (TFR) modeling approaches to more than two stress levels. See, for example, Pan and Balakrishnan* (2010), Samanta et al. (2019), Tang (2003) for generalization of the CE modeling and Madi (1993), Balakrishnan et al. (2009), Wang and Fei (2003) for generalization of the TFR modeling. See also Kundu and Ganguly (2017). However, although there have been a number of works on estimation and optimal design for a simple SSLT under TRV modeling (See Goel (1975), Bai and Chung (1992), Bai et al. (1993)), there has not been any attempt to generalize the TRV modeling approach to more than two stress levels, as far as we know. The objective of the present work is to fill this gap incorporating multiple tampering coefficients in a natural way.

Meanwhile, there have been some work establishing 'equivalence' between CE and TRV modeling, and also TRV and TFR modeling, in the context of simple SSLT (See Wang and Fei (2004)) in the sense that, for any fixed change point τ_1, the two distributions are from the same parametric family. In this work, we also make an attempt to establish such equivalence in the general SSLT framework with more than two stress levels.

The idea behind the proposed generalization of the TRV modeling approach for multiple step-stress life testing can be described as follows. As the life time passes through different stress conditions, it becomes tampered over and above the already tampered life time due to the previous stress conditions. In other words, tampering of the remaining lifetime at a stress condition takes place on the successively tampered lifetime due to the previous stress conditions. This modeling approach can be readily extended to include the specific stress levels, as in Shaked and Singpurwalla (1983), and also to model discrete and multivariate life time.

In Section 2, we introduce the TRV modeling for general multiple step-stress life testing and derive the corresponding distribution function (CDF), density function (PDF) and hazard rate. Also, the corresponding expectation and moment generating function (MGF) are derived in most general form. In particular, algebraic expressions for the CDF, the mean and the MGF, corresponding to the exponential and Weibull distributions for the baseline lifetime T under normal stress condition with stress level s_1, have been worked out. Section 3 derives some results on stochastic ordering in this framework of TRV modeling. In section 4, we investigate the equivalence between CE and TRV, and also between TFR and TRV, modeling approaches under the multiple SSLT. Some variations of this TRV modeling have been discussed in Section 5, while Section 6 ends with some concluding remarks.

2 Model Description

We consider a general SSLT with k stress levels, denoted by k-SSLT, as described in the previous section. Let T be the baseline lifetime of a unit under normal stress condition with stress level s_1. So, the new random variable $T_{TRV}^{(k)}$, representing the tampered life time under the k-SSLT, is same as T in $[0, \tau_1]$. Now, if the unit survives till time τ_1, and as the stress level is changed from s_1 to s_2 at time τ_1, it is assumed that the impact of such a change is to tamper the remaining life time by an unknown scale factor $\beta_1 < 1$, as in the simple SSLT. That is, after time τ_1, the tampered life time $T_{TRV}^{(k)}$ becomes $\tau_1 + \beta_1(T - \tau_1)$, if no further change of stress level is required (that is, the unit fails before time τ_2). However, as the stress condition is scheduled to change from stress level s_2 to s_3 at time τ_2, the once-tampered life time of the unit, if it survives till τ_2, goes through further tampering by another unknown tampering coefficient $\beta_2 < 1$ after time τ_2. Note that, as the once-tampered life time reaches τ_2, the baseline life time T under the initial normal stress condition would have reached $\tau_1 + \frac{(\tau_2 - \tau_1)}{\beta_1} = \tau_2^*$, say. Therefore, we can write the tampered life time $T_{TRV}^{(k)}$ as $\tau_1 + \beta_1(T - \tau_1)$ for T in $(\tau_1, \tau_2^*]$. By the similar argument, the twice-tampered life time $T_{TRV}^{(k)}$ after time τ_2 can be written as $\tau_2 + \beta_1\beta_2(T - \tau_2^*)$ for T in $(\tau_2^*, \tau_3^*]$, where $\tau_3^* = \tau_2^* + \frac{(\tau_3 - \tau_2)}{\beta_1\beta_2}$. Continuing this argument over the k scheduled stress conditions (for $k \geq 2$) with stress levels $s_1, s_2, \ldots, s_k$, we have the successively tampered life time given by

$$T_{TRV}^{(k)} = \begin{cases} T, & 0 \le T \le \tau_1 \\ \tau_1 + \beta_1(T - \tau_1^*), & \tau_1^* < T \le \tau_2^* \\ \tau_2 + \beta_1\beta_2(T - \tau_2^*), & \tau_2^* < T \le \tau_3^* \\ \vdots \\ \tau_{k-1} + \prod_{i=1}^{k-1} \beta_i (T - \tau_{k-1}^*), & T > \tau_{k-1}^*, \end{cases} \tag{4}$$

where $\tau_1^* = \tau_1$ and $\tau_i^* = \tau_{i-1}^* + \frac{(\tau_i - \tau_{i-1})}{\prod_{j=1}^{i-1} \beta_j}$ for $i = 2, \dots k-1$. One can write $\tau_i^* = \tau_1 + \sum_{l=2}^{i} \left(\prod_{j=1}^{l-1} \beta_j \right)^{-1} (\tau_l - \tau_{l-1})$, for $i = 2, \dots k-1$. It can be verify that, as $T_{TRV}^{(k)}$ lies between τ_{i-1} and τ_i, the baseline lifetime T takes values between τ_{i-1}^* and τ_i^*, for $i = 1, \dots k+1$, with $\tau_0 = 0$ and $\tau_{k+1} = \infty$. Note that, as $\tau_1 < \tau_2 < \dots < \tau_{k-1}$ are the time points for accelerating the stress levels on the tampered and observable life time $T_{TRV}^{(k)}$, the corresponding change points for the baseline and unobservable lifetime T are $\tau_1^* < \tau_2^* < \dots < \tau_{k-1}^*$. That is, as $T_{TRV}^{(k)}$ takes the values τ_i, the corresponding value of T is τ_i^*, for $i = 1, \dots k-1$, with $\tau_1^* = \tau_1$ and $\tau_i^* > \tau_i$, for $i = 2, \dots k-1$, as expected. In the context of TRV modeling for k-SSLT, as remarked before, these τ_i's are called the tampering times and the β_i's are called the corresponding tampering coefficients. Note that the tampering effect in a stress condition acts on the successively tampered life time due to the previous stress conditions. Therefore, the distribution of the tampered life time $T_{TRV}^{(k)}$ depends on the entire history of the SSLT experiment including the pattern of change in stress conditions.

Let $F_T(t;\theta)$, $f_T(t;\theta)$ and $\lambda_T(t;\theta)$ denote the cumulative distribution function (CDF), the probability density function (PDF) and hazard rate, respectively, for the baseline lifetime T, which may belong to any parametric or non-parametric family of distributions under normal stress condition, where θ denotes the associated model parameter(s). Clearly, $T_{TRV}^{(1)}$ represents the baseline lifetime T with distribution function $F_{T_{TRV}^{(1)}}$, or F_T. For notational ease, we suppress the dependence of the different CDF, PDF and hazard rate on θ in the following descriptions. Then, for a fixed value of k, it can be easily checked that, under the above TRV modeling assumption, the CDF, $F_{TRV}^{(k)}(\cdot)$, of $T_{TRV}^{(k)}$ is given by

$$F_{TRV}^{(k)}(t) = \begin{cases} F_T(t), & 0 < t \le \tau_1 \\ F_T\left(\tau_1^* + \dfrac{t-\tau_1}{\beta_1}\right), & \tau_1 < t \le \tau_2 \\ F_T\left(\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2}\right), & \tau_2 < t \le \tau_3 \\ \vdots & \\ F_T\left(\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right), & t > \tau_{k-1}. \end{cases}$$

Note that, although not explicitly mentioned, this distribution of $T_{TRV}^{(k)}$ given by the CDF above also depends on, besides θ, the unknown tampering coefficients $\beta_1, \cdots, \beta_{k-1}$. The corresponding PDF, $f_{TRV}^{(k)}(\cdot)$, is given by

$$f_{TRV}^{(k)}(t) = \begin{cases} f_T(t), & 0 \le t \le \tau_1 \\ \dfrac{1}{\beta_1} f_T\left(\tau_1^* + \dfrac{t-\tau_1}{\beta_1}\right), & \tau_1 < t \le \tau_2 \\ \dfrac{1}{\beta_1\beta_2} f_T\left(\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2}\right), & \tau_2 < t \le \tau_3 \\ \vdots & \\ \dfrac{1}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right), & t > \tau_{k-1}, \end{cases}$$

and the hazard rate, $\lambda_{TRV}^{(k)}(\cdot)$, is given by

$$\lambda_{TRV}^{(k)}(t) = \begin{cases} \lambda_T(t), & 0 \le t \le \tau_1 \\ \dfrac{1}{\beta_1} \lambda_T\left(\tau_1^* + \dfrac{t-\tau_1}{\beta_1}\right), & \tau_1 < t \le \tau_2 \\ \dfrac{1}{\beta_1\beta_2} \lambda_T\left(\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2}\right), & \tau_2 < t \le \tau_3 \\ \vdots & \\ \dfrac{1}{\prod_{i=1}^{k-1}\beta_i} \lambda_T\left(\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right), & t > \tau_{k-1}. \end{cases}$$

Let us now derive the expectation of $T_{TRV}^{(k)}$ for different values of k. Let us denote this expectation $E[T_{TRV}^{(k)}]$ by $E^{(k)}$ so that $E^{(1)} = E[T] = \int_0^\infty [1 - F_T(t)]dt$. For the simple SSLT with two stress conditions ($k = 2$) and one prefixed tampering time τ_1, the expectation of $T_{TRV}^{(2)}$ is

$$\begin{aligned}
E^{(2)} = E(T_{TRV}^{(2)}) &= \int_0^{\tau_1} (1 - F_{TRV}^{(2)}(t))dt + \int_{\tau_1}^{\infty} (1 - F_{TRV}^{(2)}(t))dt \\
&= \int_0^{\tau_1} (1 - F_T(t))dt + \int_{\tau_1}^{\infty} \left(1 - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right) dt \\
&= \int_0^{\infty} (1 - F_T(t))dt - \int_{\tau_1}^{\infty} (1 - F_T(t))dt + \int_{\tau_1}^{\infty} \left(1 - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right) dt \\
&= E^{(1)} - \int_{\tau_1}^{\infty} \left[F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right) - F_T(t)\right] dt.
\end{aligned}$$

It can be easily checked that $\tau_1^* + \dfrac{t-\tau_1}{\beta_1} > t$ for $t > \tau_1$, since $0 < \beta_1 < 1$. Hence, $E^{(2)} \leq E^{(1)}$. Similarly, for $k = 3$ with three stress conditions and the two tampering time points $\tau_1 < \tau_2$, we have

$$\begin{aligned}
E^{(3)} = E(T_{TRV}^{(3)}) &= \int_0^{\tau_1} (1 - F_{TRV}^{(3)}(t))dt + \int_{\tau_1}^{\tau_2} (1 - F_{TRV}^{(3)}(t))dt + \int_{\tau_2}^{\infty} (1 - F_{TRV}^{(3)}(t))dt \\
&= \int_0^{\tau_1} (1 - F_T(t))dt + \int_{\tau_1}^{\infty} \left(1 - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right) dt \\
&\quad - \int_{\tau_2}^{\infty} \left(1 - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right) dt + \int_{\tau_2}^{\infty} \left(1 - F_T\left(\tau_2^* + \frac{t-\tau_2}{\beta_1\beta_2}\right)\right) dt \\
&= E^{(2)} - \int_{\tau_2}^{\infty} \left[F_T\left(\tau_2^* + \frac{t-\tau_2}{\beta_1\beta_2}\right) - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right] dt.
\end{aligned}$$

It can easily be checked that $\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2} > \tau_1^* + \dfrac{t-\tau_1}{\beta_1}$ for $t > \tau_2$, since we also have $0 < \beta_2 < 1$. Hence, $E^{(3)} \leq E^{(2)}$. In general, the expectation for k stress conditions with the $(k-1)$ tampering time points $\tau_1 < \cdots < \tau_{k-1}$, we have

$$E^{(k)} = E(T_{TRV}^{(k)}) = E^{(k-1)} - \int_{\tau_{k-1}}^{\infty} \left[F_T\left(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) - F_T\left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right)\right] dt.$$

As before, it can easily be checked that $\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^* + \dfrac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$ for $t > \tau_{k-1}$, since we have $0 < \beta_i < 1$ for $i = 1, \cdots, k-1$. Hence, $E^{(k)} \leq E^{(k-1)}$.

In particular, suppose T follows an exponential distribution with mean θ (that is, $E^{(1)} = E[T] = \theta$). Then, for a simple SSLT with $k = 2$, the CDF,

$F^{(2)}_{TRV}(\cdot)$, of $T^{(2)}_{TRV}$ is given by

$$F^{(2)}_{TRV}(t) = \begin{cases} 1 - e^{-\frac{t}{\theta}}, & 0 \le t \le \tau_1 \\ 1 - e^{-\frac{1}{\theta}\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)}, & t > \tau_1, \end{cases}$$

with the corresponding PDF, $f^{(2)}_{TRV}(t)$, given by

$$f^{(2)}_{TRV}(t) = \begin{cases} \frac{1}{\theta} e^{-\frac{t}{\theta}}, & 0 \le t \le \tau_1 \\ \frac{1}{\theta\beta_1} e^{-\frac{1}{\theta}\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)}, & t > \tau_1, \end{cases} \tag{5}$$

and the corresponding hazard rate, $\lambda^{(2)}_{TRV}$, given by

$$\lambda^{(2)}_{TRV}(t) = \begin{cases} \frac{1}{\theta}, & 0 \le t \le \tau_1 \\ \frac{1}{\theta\beta_1}, & t > \tau_1. \end{cases} \tag{6}$$

The expectation $E^{(2)}$ of $T^{(2)}_{TRV}$ is $E^{(2)} = E^{(1)} - \theta(1-\beta_1) \times e^{-\frac{\tau_1}{\theta}}$.

Further, with $k = 3$, the expectation of $T^{(3)}_{TRV}$ is

$$E^{(3)} = E^{(2)} - \theta\beta_1(1-\beta_2) \times e^{-\frac{\tau_2^*}{\theta}},$$

where $\tau_2^* = \tau_1 + \frac{\tau_2-\tau_1}{\beta_1}$. When there are k stress conditions with the $(k-1)$ tampering time points $\tau_1 < \cdots < \tau_{k-1}$, we have

$$E^{(k)} = E(T^{(k)}_{TRV}) = E^{(k-1)} - \theta \left(\prod_{i=1}^{k-2} \beta_i\right) (1-\beta_{k-1}) \times e^{-\frac{\tau^*_{k-1}}{\theta}},$$

where $\tau^*_{k-1} = \tau_1 + \sum_{i=2}^{k-1} \left(\prod_{j=1}^{i-1} \beta_j\right)^{-1} (\tau_i - \tau_{i-1})$.

Suppose now that the baseline lifetime variable T under normal stress condition follows $Weibull(\alpha, \lambda)$ with the CDF $F_T(t) = 1 - e^{-\lambda t^{\alpha}}$ and the expectation given by

$$E^{(1)} = E(T) = \lambda^{-\frac{1}{\alpha}} \Gamma(\frac{1}{\alpha} + 1).$$

Then, the CDF and PDF for $T_{TRV}^{(2)}$ can be written as

$$F_{TRV}^{(2)}(t) = \begin{cases} 1 - e^{-\lambda\, t^{\alpha}}, & 0 \le t \le \tau_1 \\ 1 - e^{-\lambda\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha}}, & t > \tau_1 \end{cases}$$

and

$$f_{TRV}^{(2)}(t) = \begin{cases} \lambda\alpha\, t^{\alpha-1}\, e^{-\lambda t^{\alpha}}, & 0 \le t \le \tau_1 \\ \dfrac{\lambda\alpha}{\beta_1}\left(\tau_1 + \dfrac{t-\tau_1}{\beta_1}\right)^{\alpha-1} e^{-\lambda\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha}}, & t > \tau_1, \end{cases} \tag{7}$$

respectively. Therefore, the expectation of $T_{TRV}^{(2)}$ is

$$\begin{aligned} E^{(2)} &= \lambda\alpha \int_0^{\tau_1} t\, t^{\alpha-1} e^{-\lambda t^{\alpha}}\ dt + \frac{\lambda\alpha}{\beta_1}\int_{\tau_1}^{\infty} t\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha-1} e^{-\lambda\left(\tau_1+\frac{t-\tau_1}{\beta_1}\right)^{\alpha}}\ dt \\ &= \int_0^{\lambda\tau_1^{\alpha}} (\frac{z}{\lambda})^{\frac{1}{\alpha}} e^{-z}\ dz + \int_{\lambda\tau_1^{\alpha}}^{\infty}\left(\tau_1 + \beta_1\left((\frac{z}{\lambda})^{\frac{1}{\alpha}} - \tau_1\right)\right) e^{-z}\ dz \\ &= E^{(1)} - \int_{\lambda\tau_1^{\alpha}}^{\infty}\left[(1-\beta_1)\left((\frac{z}{\lambda})^{\frac{1}{\alpha}} - \tau_1\right)\right] e^{-z}\ dz. \end{aligned}$$

Further, the expectation of $T_{TRV}^{(3)}$ can be written as

$$E^{(3)} = E^{(2)} - \int_{\lambda{\tau_2^*}^{\alpha}}^{\infty}\left[\beta_1(1-\beta_2)(\frac{z}{\lambda})^{\frac{1}{\alpha}} - (1-\beta_1\beta_2)\tau_2 + (1-\beta_1)\tau_1\right] e^{-z}\ dz.$$

Similarly, for the k-SSLT with the $(k-1)$ tampering time points $\tau_1 < \cdots < \tau_{k-1}$, we have

$$\begin{aligned} E^{(k)} = E(T_{TRV}^{(k)}) \quad = \quad & E^{(k-1)} - \int_{\lambda{\tau_{k-1}^*}^{\alpha}}^{\infty}\left(\prod_{i=1}^{k-2}\beta_i\right)(1-\beta_{k-1})(\frac{z}{\lambda})^{\frac{1}{\alpha}} e^{-z}\ dz \\ & - \int_{\lambda{\tau_{k-1}^*}^{\alpha}}^{\infty}\left[\left(1 - \prod_{i=1}^{k-1}\beta_i\right)\tau_{k-1} + \left(1 - \prod_{i=1}^{k-2}\beta_i\right)\tau_{k-2}\right] e^{-z}\ dz. \end{aligned}$$

One can also derive the moment generating function (MGF) of $T_{TRV}^{(k)}$, for a fixed k, as

$$
\begin{aligned}
M^{(k)}(u) &= E(e^{u\,T^{(k)}_{TRV}}) \\
&= \int_{-\infty}^{\tau_{k-1}} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt + \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
&= \int_{-\infty}^{\tau_{k-1}} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt + \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt \\
&- \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt + \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
&= \int_{-\infty}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt - \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt \\
&+ \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
&= M^{(k-1)}(u) + \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
&- \int_{\tau_{k-1}}^{\infty} \frac{e^{u\,t}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt.
\end{aligned}
$$

After differentiating l times, for $l \geq 1$, and then evaluating at $u = 0$, we have

$$
\begin{aligned}
E\left[\left(T^{(k)}_{TRV}\right)^l\right] = E\left[\left(T^{(k-1)}_{TRV}\right)^l\right] &+ \int_{\tau_{k-1}}^{\infty} t^l \frac{1}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
&- \int_{\tau_{k-1}}^{\infty} t^l \frac{1}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau^*_{k-2} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt. \qquad (8)
\end{aligned}
$$

It is seen, through integration by parts, that the integration of the first term

$$
\int_{\tau_{k-1}}^{\infty} t^l \frac{1}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) \; dt
$$

$$
\begin{aligned}
&= -t^l \bar{F}_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right)\Bigg|_{\tau_{k-1}}^{\infty} + \int_{\tau_{k-1}}^{\infty} l\, t^{l-1} \bar{F}_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
&= \tau^l_{k-1} \bar{F}_T(\tau^*_{k-1}) + l \int_{\tau_{k-1}}^{\infty} t^{l-1} \bar{F}_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt.
\end{aligned}
$$

Similarly, the integration of the second term

$$\int_{\tau_{k-1}}^{\infty} t^l \frac{1}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right)\, dt$$

$$= \tau_{k-1}^l \bar{F}_T\left(\tau_{k-2}^* + \frac{\tau_{k-1}-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) + l\int_{\tau_{k-1}}^{\infty} t^{l-1}\bar{F}_T\left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt.$$

Noting that $\tau_{k-2}^* + \dfrac{\tau_{k-1}-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i} = \tau_{k-1}^*$ and $\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^* + \dfrac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$, for $t > \tau_{k-1}$ and since $0 < \beta_{k-1} < 1$, it can be easily checked that the second term in (8) is less than the third term in (8). Hence, $E\left[\left(T_{TRV}^{(k)}\right)^l\right] < E\left[\left(T_{TRV}^{(k-1)}\right)^l\right]$, for all $l = 1, 2, \cdots$.

In Figure 1, we plot the CDF, PDF and the hazard rate of $T_{TRV}^{(k)}$, for $k = 1, 2, 3$, having the two tampering time points $\tau_1 = 0.5 < \tau_2 = 0.65$ and the two tampering coefficients $\beta_1 = 0.40,\ \beta_2 = 0.65$ for different distributions of T. In the top panel, the figures correspond to the exponential distribution for T with mean $\theta = 1$. The middle and bottom panels have the figures for the Weibull distribution of T with scale parameter $\lambda = 1$ and shape parameter $\alpha = 1.5$ and 0.8, respectively. The solid line represents the figures for $T = T_{TRV}^{(1)}$, while the broken and dotted lines represent the figures for $T_{TRV}^{(2)}$ and $T_{TRV}^{(3)}$, respectively. Clearly, the CDF curves lie one above the other with that for $T_{TRV}^{(1)}$ lying at the bottom. The discontinuity in the PDF and hazard rate curves at τ_1 and τ_2 are also apparent. The expectations $E^{(k)}$, for $k = 1, 2, 3$, are $(1, 0.63608, 0.57772)$, $(0.90274, 0.62298, 0.61330)$ and $(1.1330, 0.67401, 0.55519)$ for the three distributions of T considered in Figure 1, respectively.

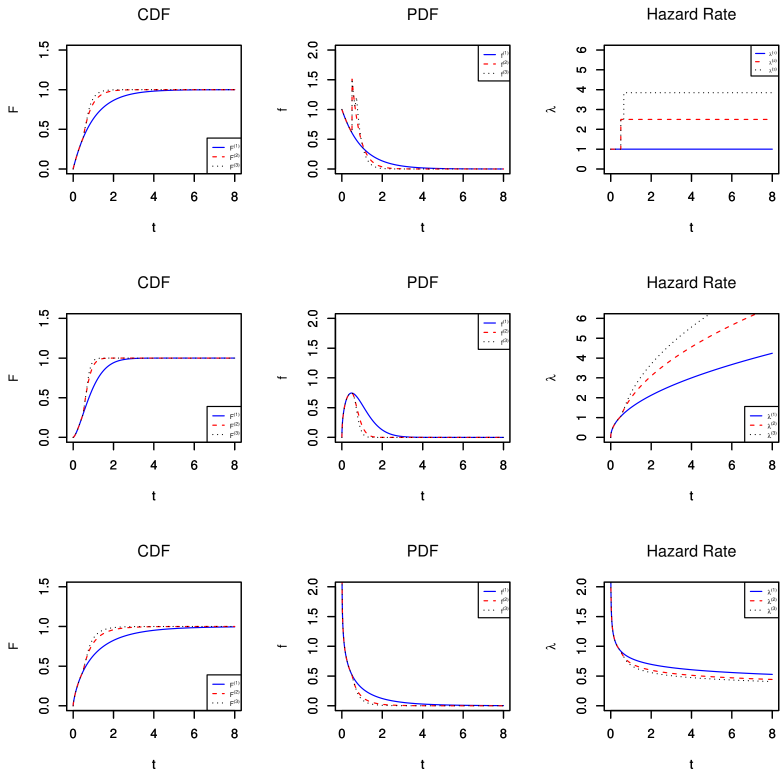

Figure 1: Plots for PDF, CDF and hazard rate of $T^{(k)}_{TRV}$, for $k = 1, 2, 3$, having the two tampering time points $\tau_1 = 0.5 < \tau_2 = 0.65$ and the two tampering coefficients $\beta_1 = 0.40,\ \beta_2 = 0.65$, when the baseline lifetime T under normal stress condition follows exponential distribution with mean $\theta = 1$ and Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\alpha = 1.5$ and 0.8.

3 Some Results on Stochastic Ordering

We can say a random variable X (or equivalently its distribution function $F_X(.)$) is greater than another random variable Y in several ways. Some common examples of partial ordering between random variables are stochastic ordering, hazard rate ordering, likelihood ratio ordering and mean time to failure (MTTF) ordering. In this section, we investigate these partial orderings for the TRV at different number of stress conditions. In other words, we explore how the lifetime random variables $T^{(k)}_{TRV}$, for $k = 1, 2, \cdots$ are ordered.

3.1 Stochastic Ordering

A random variable X is said to be stochastically larger than a random variable Y, (denoted by $\overset{st}{\geq}$) if $P(X > t) \geq P(Y > t)$ (i.e., $F_X(t) \leq F_Y(t)$), for all t. See Bergmann (1991), Dykstra et al. (1991), and Bäuerle and Müller (2006) for details. Considering the expressions of $F_{TRV}^{(k)}(\cdot)$ for different k in Section 2, we have $F_{TRV}^{(k)}(t) = F_{TRV}^{(k-1)}(t)$ for all $t \leq \tau_{k-1}$. For $t > \tau_{k-1}$, note that $\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$, since $0 < \beta_i < 1$ for all i. Hence we have $F_{TRV}^{(k)}(t) \geq F_{TRV}^{(k-1)}(t)$ for $t > \tau_{k-1}$. Therefore, the $T_{TRV}^{(k)}$'s are stochastically decreasing in k. The decreasing order of the $E^{(k)}$'s obtained in Section 2 is a consequence of this ordering.

3.2 Hazard Rate Ordering

Given two non-negative random variables X and Y with absolutely continuous CDFs, X is said to be greater than a random variable Y in the hazard rate ordering (written as $X \overset{hr}{\geq} Y$) if $\lambda_X(t) = \frac{f_X(t)}{S_X(t)} \leq \lambda_Y(t) = \frac{f_Y(t)}{S_Y(t)}$, for all $t \geq 0$. For details, one is referred to Boland et al. (1994) and Nanda and Shaked (2001).

In order to study the hazard rate ordering of the $T_{TRV}^{(k)}$'s for different k, let us consider the expression for the hazard rate of $T_{TRV}^{(k)}$ derived in Section 2. We also assume that the original life time T follows increasing failure rate (IFR) distribution. Comparing $\lambda_{TRV}^{(k)}(t)$ with $\lambda_{TRV}^{(k-1)}(t)$, we have $\lambda_{TRV}^{(k)}(t) = \lambda_{TRV}^{(k-1)}(t)$ for all $t \leq \tau_{k-1}$. For $t > \tau_{k-1}$, recall that

$$\lambda_{TRV}^{(k)}(t) = \frac{1}{\prod_{i=1}^{k-1}\beta_i}\lambda_T\left(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right)$$

and

$$\lambda_{TRV}^{(k-1)}(t) = \frac{1}{\prod_{i=1}^{k-2}\beta_i}\lambda_T\left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right).$$

Since $0 < \beta_i < 1$ for all i, $\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$ (See Section 3.1) and the distribution of T is IFR, we have $\lambda_{TRV}^{(k)}(t) \geq \lambda_{TRV}^{(k-1)}(t)$ for $t > \tau_{k-1}$. Hence, $T_{TRV}^{(k-1)}$ is larger than $T_{TRV}^{(k)}$ in the hazard rate ordering, for $k = 1, 2, \cdots$, provided the baseline lifetime T is IFR.

3.3 Likelihood Ratio Ordering

Given two random variables X and Y with PDFs (or PMFs) $f_X(\cdot)$ and $f_Y(\cdot)$, respectively, X is said to be larger than Y in the likelihood ratio order (written as $X \overset{lr}{\geq} Y$), if $\frac{f_X(t)}{f_Y(t)}$ is a non-decreasing function of t over the union of supports of X and Y. For more details, see Bapat and Kochar (1994), Ma (1998), and Yang and Zhuang (2014).

For investigating the likelihood ratio ordering of the $T_{TRV}^{(k)}$'s for different k, note the expression for the PDF of $T_{TRV}^{(k)}$ derived in Section 2. Comparing $f_{TRV}^{(k)}(t)$ with $f_{TRV}^{(k-1)}(t)$, we have $f_{TRV}^{(k)}(t) = f_{TRV}^{(k-1)}(t)$ for all $t \leq \tau_{k-1}$. For $t > \tau_{k-1}$, one can easily see that

$$\frac{f_{TRV}^{(k-1)}(t)}{f_{TRV}^{(k)}(t)} = \frac{\beta_{k-1} f_T(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i})}{f_T(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i})}.$$

It is difficult to identify all the $f_T(\cdot)$'s for which this ratio is non-decreasing in t. However, if $f_T(\cdot)$ follows an Exponential distribution, then one can prove that this ratio is non-decreasing in t, since $1/\prod_{i=1}^{k-2}\beta_i < 1/\prod_{i=1}^{k-1}\beta_i$. Therefore, when $f_T(\cdot)$ follows an Exponential distribution, the $T_{TRV}^{(k)}$'s for different k are decreasing in the likelihood ratio order.

It is well-known (See, for example, Ross(1983)) that likelihood ratio ordering implies hazard rate ordering which in turn implies stochastic ordering. Hence, clearly, when T follows an Exponential distribution, the $T_{TRV}^{(k)}$'s are decreasing in k according to all these three orderings.

3.4 Mean Time to Failure Ordering

Consider the lifetime X of an item with CDF $F(\cdot)$ in an age replacement model so that the item is replaced by a new one, whose lifetime is equal to X in distribution, at failure or at age t, whichever is earlier. Then, according to Barlow and Proschan (1965), the mean time to failure (MTTF) is defined as

$$m_X(t) \quad = \quad \frac{\int_0^t [1 - F(x)]\, dx}{F(t)}.$$

A random variable X is said to be larger than another random variable Y in the mean time to failure (MTTF) ordering (written as $X \overset{MTTF}{\geq} Y$) if $m_X(t) \geq m_Y(t)$ for all $t \geq 0$. See Kayid et al. (2013) for details.

Asha and Nair (2010) have proved that stochastic order implies MTTF order. Therefore, since the $T_{TRV}^{(k)}$'s are stochastically decreasing in k, these are also decreasing in MTTF order.

4 Equivalence Results

In this section, we explore the relationship between the TRV model with the other two models, namely, CE and TFR, under the framework of k-SSLT. Wang and Fei (2004) studied the conditions for 'equivalence' of the TRV model with the CE and TFR models under the simple SSLT (that is, 2-SSLT). As in Section 2, let $F_T(\cdot)$ be the CDF of lifetime under normal stress condition with the corresponding survival function $\bar{F}_T(\cdot)$. The survival function of the overall lifetime for the TFR model under the 2-SSLT can be obtained as

$$\bar{F}_{TFR}^{(2)}(t) = \begin{cases} \bar{F}(t), & \text{if } t \leq \tau_1 \\ \bar{F}(\tau_1)\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}, & \text{if } t > \tau_1, \end{cases} \tag{9}$$

where $\alpha_1 > 1$, while the CDF $F_{CE}^{(2)}(\cdot)$ of overall lifetime for the CE model under the 2-SSLT framework is given in Section 1. Note that the $F_1(\cdot)$ in the expression for $F_{CE}^{(2)}(\cdot)$ is the CDF of lifetime under normal stress condition and, hence, is same as $F_T(\cdot)$.

Wang and Fei (2004) have proved that the survival function $\bar{F}_{TFR}^{(2)}(\cdot)$ and that of $T_{TRV}^{(2)}$ for TRV model (See Section 2) are equivalent if and only if $F_T(\cdot)$ is an Exponential distribution, in the sense that, for any given τ_1, there exist $0 < \beta_1 < 1$ and $\alpha_1 > 1$ such that $\bar{F}_{TFR}^{(2)}(t) = \bar{F}_{TRV}^{(2)}(t)$ for all $t > 0$. They have also proved that the CDF $F_{CE}^{(2)}(t) = F_{TRV}^{(2)}(t)$ for all $t > 0$ if and only if $F_1(\cdot)$ and $F_2(\cdot)$ belong to the same scale parameter family of distributions. One such example is that $F_i(t) = F(\frac{t}{s_i})$, $i = 1, 2$, for some CDF $F(\cdot)$ (Nelson (1980)). In this section, we prove the similar equivalence results under the framework of k-SSLT for $k \geq 2$. We start by proving the results for $k = 3$ and then generalize the arguments for general k.

Theorem 1. *In a 3-SSLT framework, the TRV model is equivalent to the TFR model if and only if the baseline lifetime distribution $F_T(\cdot)$, or $F_1(\cdot)$, is exponential.*

Proof: Suppose $\bar{F}_1(t)$ is the exponential survival function given by $\bar{F}_1(t) =$

$e^{-\frac{t}{\theta}}$.

Note that the survival function of the overall lifetime for the TFR model under 3-SSLT is given as (Madi (1993))

$$\bar{F}^{(3)}_{TFR}(t) = \begin{cases} \bar{F}_1(t), & \text{if } t \leq \tau_1 \\ \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}, & \text{if } \tau_1 < t \leq \tau_2 \\ \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(\tau_2)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_2)}\right]^{\alpha_1\alpha_2}, & \text{if } t > \tau_2, \end{cases}$$

where $\alpha_i > 1$, for $i = 1, 2$. The same for the TRV model, using the expression for CDF of $T^{(3)}_{TRV}$ from Section 2, is given by

$$\bar{F}^{(3)}_{TRV}(t) = \begin{cases} \bar{F}_1(t), & \text{if } t \leq \tau_1 \\ \bar{F}_1(\tau_1^* + \frac{t-\tau_1}{\beta_1}), & \text{if } \tau_1 < t \leq \tau_2 \\ \bar{F}_1(\tau_2^* + \frac{t-\tau_2}{\beta_1\beta_2}), & \text{if } t > \tau_2, \end{cases}$$

where $0 < \beta_i < 1$, for $i = 1, 2$, are the tampering coefficients. It is already shown in Wang and Fei (2004) that $\bar{F}^{(3)}_{TFR}(t) = \bar{F}^{(3)}_{TRV}(t)$ for $t \leq \tau_2$. For $t > \tau_2$ and exponential baseline lifetime distribution, we have

$$\begin{aligned} \bar{F}^{(3)}_{TFR}(t) &= \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(\tau_2)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_2)}\right]^{\alpha_1\alpha_2} \\ &= e^{-\frac{1}{\theta}(\tau_1+\alpha_1(\tau_2-\tau_1)+\alpha_1\alpha_2(t-\tau_2))} \end{aligned} \tag{10}$$

and

$$\bar{F}^{(3)}_{TRV}(t) = \bar{F}_1\left(\tau_2^* + \frac{t-\tau_2}{\beta_1\beta_2}\right) = e^{-\frac{1}{\theta}\left(\tau_1+\frac{\tau_2-\tau_1}{\beta_1}+\frac{t-\tau_2}{\beta_1\beta_2}\right)}. \tag{11}$$

So, taking $\alpha_i = 1/\beta_i$, for $i = 1, 2$, results in the equivalence between the TRV and TFR models.

The proof of the necessary part is similar to Wang and Fei (2004). Suppose $\bar{F}^{(3)}_{TRV}(t) = \bar{F}^{(3)}_{TFR}(t)$, for all $t > 0$. We need to prove that $F_1(\cdot)$ is an exponential distribution function. For $t > \tau_2$, we have

$$\bar{F}_1(\tau_1 + \frac{\tau_2-\tau_1}{\beta_1} + \frac{t-\tau_2}{\beta_1\beta_2}) = \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(\tau_2)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_2)}\right]^{\alpha_1\alpha_2},$$

for any choice of $\tau_2 > \tau_1 > 0$. Taking logarithm on both sides, as in Wang and

Fei (2004), we get

$$\log \bar{F}_1(\tau_1 + \frac{\tau_2 - \tau_1}{\beta_1} + \frac{t - \tau_2}{\beta_1\beta_2}) = \log\left(\bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(\tau_2)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}\right) + \alpha_1\alpha_2 \log\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_2)}\right].$$

Then, taking derivative with respect to t on both sides, we have

$$\frac{1}{\beta_1\beta_2}\frac{f_1\left(\tau_1 + \frac{\tau_2-\tau_1}{\beta_1} + \frac{t-\tau_2}{\prod_{i=1}^{2}\beta_i}\right)}{\bar{F}_1\left(\tau_1 + \frac{\tau_2-\tau_1}{\beta_1} + \frac{t-\tau_2}{\prod_{i=1}^{2}\beta_i}\right)} = \alpha_1\alpha_2\left[\frac{f_1(t)}{\bar{F}_1(t)}\right], \tag{12}$$

where $f_1(\cdot)$ is the baseline lifetime density corresponding to $F_1(\cdot)$. Now letting $t \to \tau_2$ and $\tau_2 \to \tau_1$ we get,

$$\frac{1}{\beta_1\beta_2}\frac{f_1(\tau_1)}{\bar{F}_1(\tau_1)} = \alpha_1\alpha_2\frac{f_1(\tau_1)}{\bar{F}_1(\tau_1)}.$$

Therefore, because of the arbitrariness of τ_1 and τ_2, we have $\frac{1}{\beta_1\beta_2} = \alpha_1\alpha_2$.

Now, letting $\tau_2 \to \tau_1$ in (12), we have

$$\frac{1}{\beta_1\beta_2}\frac{f_1\left(\tau_1 + \frac{t-\tau_1}{\prod_{i=1}^{2}\beta_i}\right)}{\bar{F}_1\left(\tau_1 + \frac{t-\tau_1}{\prod_{i=1}^{2}\beta_i}\right)} = \alpha_1\alpha_2\left[\frac{f_1(t)}{\bar{F}_1(t)}\right].$$

Since $\frac{1}{\beta_1\beta_2} = \alpha_1\alpha_2$, we have

$$\frac{f_1(\tau_1 + \frac{t-\tau_1}{\beta_1\beta_2})}{\bar{F}_1(\tau_1 + \frac{t-\tau_1}{\beta_1\beta_2})} = \frac{f_1(t)}{\bar{F}_1(t)}.$$

Now letting $\tau_1 \to 0$, we get

$$\frac{f_1\left(\frac{t}{\beta_1\beta_2}\right)}{\bar{F}_1\left(\frac{t}{\beta_1\beta_2}\right)} = \frac{f_1(t)}{\bar{F}_1(t)}. \tag{13}$$

Writing $\frac{t}{\beta_1\beta_2} = y$, we have $t = \beta_1\beta_2 y$. So, from (13),

$$\frac{f_1(y)}{\bar{F}_1(y)} = \frac{f_1\left(\beta_1\beta_2\, t\right)}{\bar{F}_1\left(\beta_1\beta_2\, t\right)}. \tag{14}$$

Using (14) n times repeatedly yields

$$\frac{f_1(y)}{\bar{F}_1(y)} = \frac{f_1\left(\beta_1\beta_2\, t\right)}{\bar{F}_1\left(\beta_1\beta_2\, t\right)} = \cdots = \frac{f_1\left(\beta_1^n\beta_2^n\, t\right)}{\bar{F}_1\left(\beta_1^n\beta_2^n\, t\right)}. \tag{15}$$

Letting $n \to \infty$, (15) becomes $\frac{f_1(y)}{\bar{F}_1(y)} = \lim_{n\to\infty} \frac{f_1(\beta_1^n\beta_2^n\, t)}{\bar{F}_1\left(\beta_1^n\beta_2^n\, t\right)} = \frac{f_1(0)}{\bar{F}_1(0)} = f_1(0)$. That is, the failure rate of the distribution $F_1(t)$ is the constant $f_1(0)$. Furthermore, we have $\bar{F}_1(y) = e^{-\int_0^y f_1(0)\ dx} = e^{-f_1(0)y}$; that is to say that the distribution function F_1 is exponential.

Theorem 2. *In a k-SSLT framework, the TRV model is equivalent to the TFR model if and only if the baseline lifetime distribution $F_T(\cdot)$, or $F_1(\cdot)$, is exponential.*

Proof: Let us first assume that $F_1(\cdot)$ is exponential with mean θ. Note that the survival function of TFR model under the k-SSLT framework is given as (Madi (1993))

$$\bar{F}^{(k)}_{TFR}(t) = \begin{cases} \bar{F}_1(t), & \text{if } t \le \tau_1 \\ \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}, & \text{if } \tau_1 < t \le \tau_2 \\ \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(\tau_2)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1}\left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_2)}\right]^{\alpha_1\alpha_2}, & \text{if } \tau_2 < t \le \tau_3, \\ \vdots & \\ \bar{F}_1(\tau_1)\left[\frac{\bar{F}_1(\tau_2)}{\bar{F}_1(\tau_1)}\right]^{\alpha_1} \times \cdots \times \left[\frac{\bar{F}_1(t)}{\bar{F}_1(\tau_{k-1})}\right]^{\prod_{j=1}^{k-1}\alpha_j}, & t > \tau_{k-1}. \end{cases}$$

For exponential baseline lifetime distribution, the survival function, for $\tau_{i-1} < t \le \tau_i$, reduces to

$$\bar{F}^{(k)}_{TFR}(t) = exp\left(-\frac{1}{\theta}\left(\tau_1 + \alpha_1(\tau_2 - \tau_1) + \cdots + \prod_{j=1}^{i-1}\alpha_j(t - \tau_{i-1})\right)\right),$$

for $i = 2, \cdots k$, with $\tau_k = \infty$. Clearly, for $t \le \tau_1$, $\bar{F}^{(k)}_{TFR}(t) = \bar{F}_1(t) = e^{-\frac{t}{\theta}}$. From Section 2, the survival function of the TRV model uder the k-SSLT framework and exponential baseline lifetime distribution, when $\tau_{i-1} < t \le \tau_i$, is

$$\begin{aligned}\bar{F}^{(k)}_{TRV}(t) &= e^{-\frac{1}{\theta}\left(\tau^*_{i-1}+\frac{t-\tau_{i-1}}{\prod_{j=1}^{i-1}\beta_j}\right)} \\ &= e^{-\frac{1}{\theta}\left(\tau_1+\frac{\tau_2-\tau_1}{\beta_1}+\cdots+\frac{t-\tau_{i-1}}{\prod_{j=1}^{i-1}\beta_j}\right)},\end{aligned}$$

for $i = 2, \cdots k$, with $\tau_k = \infty$. Again, for $t \le \tau_1$, $\bar{F}^{(k)}_{TRV}(t) = \bar{F}_1(t) = e^{-\frac{t}{\theta}}$.

Hence, it can be easily checked that, with $\alpha_i = \frac{1}{\beta_1}$, for $i = 1, \cdots, k-1$, the two survival functions $\bar{F}^{(k)}_{TFR}(t)$ and $\bar{F}^{(k)}_{TRV}(t)$ are equal for all t. Hence, the TFR and TRV models are equivalent if the baseline lifetime distribution is exponential.

Conversely, let us suppose that $\bar{F}^{(k)}_{TFR}(t) = \bar{F}^{(k)}_{TRV}(t)$ for all $t > 0$. In particular, for $t > \tau_{k-1}$, we have

$$\bar{F}_1\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) = [\bar{F}_1(\tau_1)]^{1-\alpha_1}\left(\prod_{i=2}^{k-1}[\bar{F}_1(\tau_i)]^{(1-\alpha_i)\prod_{j=1}^{i-1}\alpha_j}\right)\left[\bar{F}_1(t)\right]^{\prod_{j=1}^{k-1}\alpha_j}.$$

Taking logarithm on both sides, as before for $k = 3$, and then taking derivative with respect to t, we get

$$\frac{1}{\prod_{i=1}^{k-1}\beta_i}\frac{f_1\left(\tau^*_{k-1}+\frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right)}{\bar{F}_1\left(\tau^*_{k-1}+\frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right)} = \left[\prod_{i=1}^{k-1}\alpha_i\right]\left[\frac{f_1(t)}{\bar{F}_1(t)}\right]. \tag{16}$$

As before, since the τ_i's are arbitrarily fixed, letting $t \to \tau_{k-1}$ and $\tau_{k-1} \to \tau_{k-2}, \cdots, \tau_2 \to \tau_1$, we get

$$\frac{1}{\prod_{i=1}^{k-1}\beta_i}\frac{f_1(\tau_1)}{\bar{F}_1(\tau_1)} = \left[\prod_{i=1}^{k-1}\alpha_i\right]\frac{f_1(\tau_1)}{\bar{F}_1(\tau_1)}$$

Hence, because of the arbitrariness of τ_1, we have

$$\frac{1}{\prod_{i=1}^{k-1}\beta_i} = \prod_{i=1}^{k-1}\alpha_i.$$

Now letting $\tau_{k-1} \to \tau_{k-2}, \cdots, \tau_2 \to \tau_1$ and $\tau_1 \to 0$ in (16) and using the result $\left[\prod_{i=1}^{k-1}\beta_i\right]^{-1} = \prod_{i=1}^{k-1}\alpha_i$, we have

$$\frac{f_1\left(\frac{t}{\prod_{i=1}^{k-1}\beta_i}\right)}{\bar{F}_1\left(\frac{t}{\prod_{i=1}^{k-1}\beta_i}\right)} = \frac{f_1(t)}{\bar{F}_1(t)}.$$

Now, as before, assuming $\frac{t}{\prod_{i=1}^{k-1}\beta_i} = y$ and using the same argument as before for $k = 3$, it can be proved that the distribution function F_1 is exponential.

Theorem 3. *In a k-SSLT framework, the TRV model is equivalent to the CE model if and only if the marginal lifetime distributions at the k stress levels belong to the same scale parametric family.*

Proof: Let us write $F_i(\cdot)$ for the marginal lifetime distribution under the i^{th} stress condition, for $i = 1, \cdots, k$, under the k-SSLT framework. Note that the CDF of the CE model under the k-SSLT framework is given as (Balakrishnan et al. (2009), Samanta et al. (2019))

$$F_{CE}^{(k)}(t) = \begin{cases} F_1(t), & \text{if } t \leq \tau_1 \\ F_2(t-\tau_1+h_1), & \text{if } \tau_1 < t \leq \tau_2 \\ F_3(t-\tau_2+h_2), & \text{if } \tau_2 < t \leq \tau_3, \\ \vdots & \\ F_k(t-\tau_{k-1}+h_{k-1}), & t > \tau_{k-1}, \end{cases}$$

where h_i, $i = 1, \cdots, k-1$, can be successively obtained by solving

$$\begin{aligned} F_2(h_1) &= F_1(\tau_1), \\ F_3(h_2) &= F_2(\tau_2-\tau_1+h_1), \\ &\vdots \\ F_k(h_{k-1}) &= F_{k-1}(\tau_{k-1}-\tau_{k-2}+h_{k-2}). \end{aligned}$$

Let us first assume that the marginal lifetime distributions under the k stress levels belong to the same scale parametric family so that $F_i(t)$ can be written as $G(\frac{t}{\eta_i})$, where $G(\cdot)$ is some distribution function and η_i is a scale parameter, for $i = 1, \cdots, k$. Let us define $\beta_1, \beta_2, \cdots, \beta_{k-1}$ in a way such that $\prod_{j=1}^{i}\beta_j = \frac{\eta_{i+1}}{\eta_1}$, for $i = 1, \cdots, k-1$, that is, $\beta_i = \frac{\eta_{i+1}}{\eta_i}$, for $i = 1, \cdots, k-1$.

By definition, we have $F_{CE}^{(k)}(t) = F_{TRV}^{(k)}(t)$ for $t \leq \tau_1$. Wang and Fei (2004) have proved this equality for $\tau_1 < t \leq \tau_2$; however, we reproduce the proof in the following for the better understanding of the generalization. For $\tau_1 < t \leq \tau_2$, we have

$$\begin{aligned} F_{TRV}^{(k)}(t) &= F_1\left(\tau_1 + \frac{\eta_1}{\eta_2}(t-\tau_1)\right) \\ &= G\left(\frac{\tau_1 + \frac{\eta_1}{\eta_2}(t-\tau_1)}{\eta_1}\right) \\ &= G\left(\frac{\frac{\eta_2}{\eta_1}\tau_1 + (t-\tau_1)}{\eta_2}\right) \\ &= F_2\left(t - \tau_1 + \frac{\eta_2}{\eta_1}\tau_1\right) \\ &= F_{CE}^{(k)}(t), \quad \text{with } h_1 = \frac{\eta_2}{\eta_1}\tau_1, \end{aligned}$$

since h_1 satisfies $F_2(h_1) = F_1(\tau_1)$, or $G\left(\frac{\frac{\eta_2}{\eta_1}\tau_1}{\eta_2}\right) = G\left(\frac{\tau_1}{\eta_1}\right)$. Similarly, for $\tau_2 < t \leq \tau_3$, we have

$$\begin{aligned} F_{TRV}^{(k)}(t) &= F_1\left(\tau_2^* + \frac{\eta_1}{\eta_3}(t-\tau_2)\right) \\ &= G\left(\frac{\tau_2^* + \frac{\eta_1}{\eta_3}(t-\tau_2)}{\eta_1}\right) \\ &= G\left(\frac{\frac{\eta_3}{\eta_1}\tau_2^* + (t-\tau_2)}{\eta_3}\right) \\ &= F_3\left(t - \tau_2 + \frac{\eta_3}{\eta_1}\tau_2^*\right) \\ &= F_{CE}^{(k)}(t), \quad \text{with } h_2 = \frac{\eta_3}{\eta_1}\tau_2^*. \end{aligned}$$

We need to verify that h_2 satisfies $F_3(h_2) = F_2(\tau_2 - \tau_1 + h_1)$ or $G\left(\frac{\frac{\eta_3}{\eta_1}\tau_2^*}{\eta_3}\right) = G\left(\frac{\tau_2 - \tau_1 + \frac{\eta_2}{\eta_1}\tau_1}{\eta_2}\right)$, or $\frac{\tau_2^*}{\eta_1} = \frac{\tau_1}{\eta_1} + \frac{\tau_2 - \tau_1}{\eta_2}$. Clearly, $\frac{\tau_2^*}{\eta_1} = \frac{\tau_1 + \frac{\eta_1}{\eta_2}(\tau_2 - \tau_1)}{\eta_1} = \frac{\tau_1}{\eta_1} + \frac{\tau_2 - \tau_1}{\eta_2}$. In general, when $\tau_{i-1} < t \leq \tau_i$, for $i = 2, \cdots, k$ with $\tau_k = \infty$, we have

$$\begin{aligned}
F^{(k)}_{TRV}(t) &= F_1\left(\tau^*_{i-1} + \frac{\eta_1}{\eta_i}(t-\tau_{i-1})\right) \\
&= G\left(\frac{\tau^*_{i-1} + \frac{\eta_1}{\eta_i}(t-\tau_{i-1})}{\eta_1}\right) \\
&= G\left(\frac{\frac{\eta_i}{\eta_1}\tau^*_{i-1} + (t-\tau_{i-1})}{\eta_i}\right) \\
&= F_i\left(t-\tau_{i-1} + \frac{\eta_i}{\eta_1}\tau^*_{i-1}\right) \\
&= F^{(k)}_{CE}(t), \quad \text{with } h_{i-1} = \frac{\eta_i}{\eta_1}\tau^*_{i-1}.
\end{aligned}$$

We need to verify that h_{i-1} satisfies $F_i(h_{i-1}) = F_{i-1}(\tau_{i-1} - \tau_{i-2} + h_{i-2})$ with $h_{i-2} = \frac{\eta_{i-1}}{\eta_1}\tau^*_{i-2}$. That is, to verify

$$F_i\left(\frac{\eta_i}{\eta_1}\left(\tau^*_{i-2} + \frac{\eta_1}{\eta_{i-1}}(\tau_{i-1}-\tau_{i-2})\right)\right) = F_{i-1}\left(\frac{\eta_{i-1}}{\eta_1}\tau^*_{i-2} + (\tau_{i-1}-\tau_{i-2})\right),$$

or

$$G\left(\frac{\frac{\eta_i}{\eta_1}\left(\tau^*_{i-2} + \frac{\eta_1}{\eta_{i-1}}(\tau_{i-1}-\tau_{i-2})\right)}{\eta_i}\right) = G\left(\frac{\frac{\eta_{i-1}}{\eta_1}\tau^*_{i-2} + (\tau_{i-1}-\tau_{i-2})}{\eta_{i-1}}\right),$$

or

$$\frac{\tau^*_{i-2}}{\eta_1} + \frac{\tau_{i-1}-\tau_{i-2}}{\eta_{i-1}} = \frac{\tau^*_{i-2}}{\eta_1} + \frac{\tau_{i-1}-\tau_{i-2}}{\eta_{i-1}},$$

which is an identity. Hence, the TRV and the CE models are equivalent.

Conversely, suppose $F^{(k)}_{TRV}(t)$ and $F^{(k)}_{CE}(t)$ are equivalent for all $t > 0$. Then, when $\tau_{i-1} < t \leq \tau_i$ for $i = 1, \cdots, k$, with $\tau_0 = 0$ and $\tau_k = \infty$, we have $F_1\left(\tau^*_{i-1} + \frac{t-\tau_{i-1}}{\prod_{j=1}^{i-1}\beta_j}\right) = F_i\left(t - \tau_{i-1} + h_{i-1}\right)$. Since the τ_i's are arbitrary, letting $\tau_{i-1} \to \tau_{i-2}, \cdots, \tau_1 \to 0$, we have $\tau^*_{i-1} \to 0$ and $h_{i-1} \to 0$, using the definition of $F^{(k)}_{CE}(\cdot)$. Hence, $F_1\left(\frac{t}{\prod_{j=1}^{i-1}\beta_j}\right) = F_i(t)$. Therefore, $F_i(\cdot)$ and $F_1(\cdot)$ belong to the same scale parametric family of distributions, for $i = 2, \cdots, k$.

In the context of this equivalence result, it is relevant to mention the work of Shaked and Singpurwalla (1983) who have considered a CE model under k-SSLT with each $F_i(\dot{)}$ belonging to the same scale parameter family, which can be verified to be the TRV model similar to the one described in Section 2.

5 Other Modeling Approaches

In this section, we discuss some variations of the TRV modeling approach of Section 2 with different domains of application.

5.1 Trend Modeling

Note that the tampering coefficients β_i's of the TRV modeling of Section 2 implicitly depend on the relative magnitudes of the different stress conditions with respect to the normal stress condition. The objective of trend modeling is to explicitly model these β_i's as functions of the stress values $s_1, \cdots, s_k$ so that one can assess the extent of tampering for a particular stress value. For convenience, let us transform the stress values to $s_i' = s_i - s_1$, for $i = 1, \cdots, k$, so that s_1' is equal to 0 representing the normal stress condition. The other transformed s_i''s now represent the additional stress values over the normal stress condition.

A simple choice for modeling the tampering coefficient $\beta(s')$ at the transformed stress value $s' = s - s_1$ is $\beta(s') = \exp(-\beta s')$, with $\beta > 0$, $s \geq s_1$, so that $\beta(s')$ lies between 0 and 1 with $\beta(0) = 1$, as desired. Also note that, by this trend modeling, the tampering effect is more adverse (that is, $\beta(s')$ moves further away from 1) with accelerated stress conditions (that is, increasing values of s'), which is usually desirable. Nevertheless, the TRV model of Section 2 can now be written with $\beta_i = \beta(s_i') = \exp(-\beta s_i')$ involving only the single parameter β for the tampering coefficients. This trend modeling allows us to assess the tampering effect at any stress value s which is not in the domain of the stress conditions used for the k-SSLT under study.

5.2 Discrete Life Time

Suppose the original life time T under normal stress condition follows a discrete distribution with mass points $0 \leq x_1 < x_2 < \cdots$ having mass $p_1, p_2, \cdots$ (such that $\sum_{i=1}^{\infty} p_i = 1$) and discrete hazards $\lambda_1, \lambda_2, \cdots$ (such that $0 \leq \lambda_i \leq 1$ for all i). Consider the TRV modeling approach of Section 2 to model the distribution of $T_{TRV}^{(k)}$ when T is discrete. Note that the tampering time points τ_i's need not belong to the support $\{x_1, x_2 \cdots\}$ of T.

As in Section 2, we define

$$T_{TRV}^{(2)} = \begin{cases} T, & 0 \le T \le \tau_1 \\ \tau_1 + \beta_1(T - \tau_1^*), & T > \tau_1 \end{cases}$$

Clearly, the support of $T_{TRV}^{(2)}$ is not the same as that of T. Writing $i_1 = max\{i : x_i \le \tau_1\}$, the support of $T_{TRV}^{(2)}$ becomes $\{x_i, i = 1, 2, \cdots i_1\} \cup \{\tau_1 + \beta_1(x_i - \tau_1), i = i_1 + 1, i_1 + 2, \cdots\}$. As before, $\tau_2^* = \tau_1 + \beta_1^{-1}(\tau_2 - \tau_1)$. Writing $i_2 = max\{i : x_i \le \tau_2^*\}$, the support of $T_{TRV}^{(3)}$ becomes $\{x_i, i = 1, 2, \cdots i_1\} \cup \{\tau_1 + \beta_1(x_i - \tau_1^*), i = i_1 + 1, \cdots i_2\} \cup \{\tau_2 + \beta_1\beta_2(x_i - \tau_2^*), i = i_2 + 1, i_2 + 2, \cdots\}$, where $\tau_1^* = \tau_1$.

In general, for k stress conditions with $(k-1)$ tampering time points $\tau_1 < \tau_2 \cdots < \tau_{k-1}$, define $\tau_l^* = \tau_{l-1}^* + (\beta_1 \cdots \beta_{l-1})^{-1}(\tau_l - \tau_{l-1})$ for $l = 2, \cdots, k-1$, and $i_l = max\{i : x_i \le \tau_l^*\}$, for $l = 1, \cdots, k-1$. Then, the support of $T_{TRV}^{(k)}$ can be written as $\{x_i, i = 1, 2, \cdots i_1\} \cup \{\tau_1 + \beta_1(x_i - \tau_1^*), i = i_1 + 1, \cdots i_2\} \cup \cdots \cup \{\tau_{k-1} + (\prod_{i=1}^{k-1} \beta_i)(x_i - \tau_{k-1}^*), i = i_{k-1} + 1, i_{k-1} + 2, \cdots\}$. Note that the masses at these mass points remain the same as $p_1, p_2, \cdots$ with the same discrete hazards $\lambda_1, \lambda_2, \cdots$. The tampering effect is only on the mass points which are reduced from the original ones.

There may be some inherent difficulty in the CE and TFR modeling approach for discrete T. In the CE approach the non-uniqueness of $F_T^{-1}(\cdot)$ may be a problem. In the TFR approach, evaluating the discrete hazard will have to deal with the constraint that these lie between 0 and 1.

5.3 Bivariate Life Time

Now we discuss the TRV modeling approach for bivariate lifetime. Let us denote the bivariate lifetime of a single unit under normal stress condition by $T = (T_1, T_2)$ with the corresponding joint CDF given by $F_T(t_1, t_2)$ for $t_1, t_2 \ge 0$. Consider first the simple SSLT with a single tampering time point τ_1 with the corresponding tampered life times denoted by $T_{TRV}^{(2)} = (T_{TRV,1}^{(2)}, T_{TRV,2}^{(2)})$. Note that, in this case, there are two different tampering coefficients, $0 < \beta_1 < 1$ and $0 < \gamma_1 < 1$, say, acting on T_1 and T_2 to result in the tampered random variables $T_{TRV,1}^{(2)}$ and $T_{TRV,2}^{(2)}$, respectively. This, therefore, allows different amounts of tampering due to a single stress condition in the two component lifetimes. Then,

$T^{(2)}_{TRV,1}$ and $T^{(2)}_{TRV,2}$ are defined as

$$T^{(2)}_{TRV,1} = \begin{cases} T_1, & 0 \le T_1 \le \tau_1 \\ \tau_1 + \beta_1(T_1 - \tau_1), & T_1 > \tau_1 \end{cases}$$

and

$$T^{(2)}_{TRV,2} = \begin{cases} T_2, & 0 \le T_2 \le \tau_1 \\ \tau_1 + \gamma_1(T_2 - \tau_1), & T_2 > \tau_1, \end{cases}$$

respectively. Then, proceeding as in Section 2, the joint CDF of $(T^{(2)}_{TRV,1}, T^{(2)}_{TRV,2})$ can be obtained as

$$F^{(2)}_{TRV}(t_1, t_2) = \begin{cases} F_T(t_1, t_2), & t_1 \le \tau_1, t_2 \le \tau_1 \\ F_T\left(t_1, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 \le \tau_1, t_2 > \tau_1 \\ F_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, t_2\right), & t_1 > \tau_1, t_2 \le \tau_1 \\ F_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 > \tau_1, t_2 > \tau_1. \end{cases}$$

The corresponding joint PDF is given by

$$f^{(2)}_{TRV}(t_1, t_2) = \begin{cases} f_T(t_1, t_2), & t_1 \le \tau_1, t_2 \le \tau_1 \\ \dfrac{1}{\gamma_1} f_T\left(t_1, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 \le \tau_1, t_2 > \tau_1 \\ \dfrac{1}{\beta_1} f_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, t_2\right), & t_1 > \tau_1, t_2 \le \tau_1 \\ \dfrac{1}{\beta_1 \gamma_1} f_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 > \tau_1, t_2 > \tau_1. \end{cases}$$

Similarly, in general, for k stress conditions with $(k-1)$ tampering time points $0 < \tau_1 < \cdots < \tau_{k-1}$ and the $(k-1)$ pairs of tampering coefficients $(\beta_1, \gamma_1), \cdots, (\beta_{k-1}, \gamma_{k-1})$, the joint CDF of $T^{(k)}_{TRV} = (T^{(k)}_{TRV,1}, T^{(k)}_{TRV,2})$ can be written as

$$
F_{TRV}^{(k)}(t_1,t_2) = \begin{cases}
F_T(t_1,t_2), & t_1 \le \tau_1, t_2 \le \tau_1 \\
F_T\left(t_1, \tau_{j-1,2}^* + \dfrac{t_2-\tau_{j-1}}{\prod_{l=1}^{j-1}\gamma_l}\right), & t_1 \le \tau_1, \tau_{j-1} < t_2 \le \tau_j \\
F_T\left(t_1, \tau_{k-1,2}^* + \dfrac{t_2-\tau_{k-1}}{\prod_{l=1}^{k-1}\gamma_l}\right), & t_1 \le \tau_1, t_2 > \tau_{k-1} \\
F_T\left(\tau_{i-1,1}^* + \dfrac{t_1-\tau_{i-1}}{\prod_{l=1}^{i-1}\beta_l}, t_2\right), & \tau_{i-1} < t_1 \le \tau_i, t_2 \le \tau_1 \\
F_T\left(\tau_{k-1,1}^* + \dfrac{t_1-\tau_{k-1}}{\prod_{l=1}^{k-1}\beta_l}, t_2\right), & t_1 > \tau_{k-1}, t_2 \le \tau_1 \\
F_T\left(\tau_{i-1,1}^* + \dfrac{t_1-\tau_{i-1}}{\prod_{l=1}^{i-1}\beta_l}, \tau_{j-1,2}^* + \dfrac{t_2-\tau_{j-1}}{\prod_{l=1}^{j-1}\gamma_l}\right), & \tau_{i-1} < t_1 \le \tau_i, \tau_{j-1} < t_2 \le \tau_j \\
F_T\left(\tau_{i-1,1}^* + \dfrac{t_1-\tau_{i-1}}{\prod_{l=1}^{i-1}\beta_l}, \tau_{k-1,2}^* + \dfrac{t_2-\tau_{k-1}}{\prod_{l=1}^{k-1}\gamma_l}\right), & \tau_{i-1} < t_1 \le \tau_i, t_2 > \tau_{k-1} \\
F_T\left(\tau_{k-1,1}^* + \dfrac{t_1-\tau_{k-1}}{\prod_{l=1}^{k-1}\beta_l}, \tau_{j-1,2}^* + \dfrac{t_2-\tau_{j-1}}{\prod_{l=1}^{j-1}\gamma_l}\right), & t_1 > \tau_{k-1}, \tau_{j-1} < t_2 \le \tau_j \\
F_T\left(\tau_{k-1,1}^* + \dfrac{t_1-\tau_{k-1}}{\prod_{l=1}^{k-1}\beta_l}, \tau_{k-1,2}^* + \dfrac{t_2-\tau_{k-1}}{\prod_{l=1}^{k-1}\gamma_l}\right), & t_1 > \tau_{k-1}, t_2 > \tau_{k-1},
\end{cases}
$$

for $i, j = 2, \cdots, k-1$, where $\tau_{i,1}^* = \tau_{i-1,1}^* + (\prod_{l=1}^{i-1}\beta_l)^{-1}(\tau_i - \tau_{i-1})$ and $\tau_{i,2}^* = \tau_{i-1,2}^* + (\prod_{l=1}^{i-1}\gamma_l)^{-1}(\tau_i - \tau_{i-1})$ are the tampered version of τ_i after passing through i number of stress conditions as manifested in the first and second component of $T_{TRV}^{(k)}$, for $i = 2, \cdots, k-1$, with $\tau_{i,1}^* = \tau_{1,2}^* = \tau_1$.

This bivariate TRV modeling can be naturally extended to model multivariate lifetimes with more than two components. However, the notation becomes increasingly complicated. As for the discrete life, there are some inherent difficulties in generalizing the CE and TFR approaches to model multivariate life time as well. While for the CE approach, the inverse of the bivariate CDF $F_T(\cdot,\cdot)$ creates the difficulty, the TFR approach has to deal with the absence of a unique definition of multivariate hazard rate.

6 Conclusion

In this work, we have developed the TRV modeling approach for multiple stress conditions with the initial stress condition being the normal one and then the subsequent stress conditions being increasingly severe (that is, accelerated in its true meaning). From the development of the model, it is clear that the initial stress condition does not need to be the normal one and also the subsequent stress conditions need not be increasingly severe. If, after any tampering time point, the changed stress condition is less severe (or, more favourable to life) than the previous stress condition, the corresponding tampering coefficient will be greater than unity instead of lying between 0 and 1. Therefore, in general, one does not have to restrict the tampering coefficients between 0 and 1 while analyzing data, letting it dictate the impact of different stress conditions.

The principle behind the TRV modeling approach is simple and appealing, possibly compared to the other two approaches, namely CE and TFR modeling. Moreover, this modeling seems to be more flexible to generalize for discrete and multivariate lifetime, as discussed at the end of Sections 5.2 and 5.3, respectively. Also, simulation of a tampered lifetime is very simple by its algebraic expression in terms of the life time under normal stress condition using the tampering time points and the tampering coefficients. So, whenever the lifetime under normal stress condition can be simulated, the corresponding simulated tampered lifetime can be immediately obtained. In general, there can be variability in the amount of tampering, or the impact of different stress conditions, over the different individuals. We plan to address this issue in the Bayesian paradigm by considering the tampering coefficients as random effects.

References

Bai, D. and Chung, S. (1992). Optimal design of partially accelerated life tests for the exponential distribution under type-i censoring. *IEEE Transactions on Reliability*, 41(3):400–406.

Bai, D. S., Chung, S., and Chun, Y. (1993). Optimal design of partially accelerated life tests for the lognormal distribution under type i censoring. *Reliability Engineering & System Safety*, 40(1):85–92.

Balakrishnan, N., Beutner, E., and Kateri, M. (2009). Order restricted inference for exponential step-stress models. *IEEE Transactions on Reliability*, 58(1):132–142.

Bapat, R. and Kochar, S. C. (1994). On likelihood-ratio ordering of order statistics. *Linear Algebra and Its Applications*, 199:281–291.

Barlow, R. and Proschan, F. (1965). 1996. *Mathematical theory of reliability. SIAM (Classics in applied mathematics).*

Bäuerle, N. and Müller, A. (2006). Stochastic orders and risk measures: consistency and bounds. *Insurance: Mathematics and Economics*, 38(1):132–148.

Bergmann, R. (1991). Stochastic orders and their application to a unified approach to various concepts of dependence and association. *Lecture Notes-Monograph Series*, pages 48–73.

Bhattacharyya, G. and Soejoeti, Z. (1989). A tampered failure rate model for step-stress accelerated life test. *Communications in statistics-Theory and methods*, 18(5):1627–1643.

Boland, P. J., El-Neweihi, E., and Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics. *Journal of Applied Probability*, 31(1):180–192.

DeGroot, M. H. and Goel, P. K. (1979). Bayesian estimation and optimal designs in partially accelerated life testing. *Naval research logistics quarterly*, 26(2):223–235.

Dykstra, R., Kochar, S., Robertson, T., et al. (1991). Statistical inference for uniform stochastic ordering in several populations. *The Annals of Statistics*, 19(2):870–888.

Goel, P. K. (1971). Some estimation problems in the study of tampered random variables. *Technical Report no. 50, Department of Statistics,Carnegie-Mellon University, Pittsburgh, Pennsylvania*, 1971.

Goel, P. K. (1975). Consistency and asymptotic normality of maximum likelyhood estimators. *Scandinavian Actuarial Journal*, 1975(2):109–118.

Kundu, D. and Ganguly, A. (2017). *Analysis of Step-stress Models: Existing Results and Some Recent Developments*. Academic Press.

Ma, C. (1998). Likelihood ratio ordering of order statistics. *Journal of statistical planning and inference*, 70(2):255–261.

Madi, M. T. (1993). Multiple step-stress accelerated life test: the tampered failure rate model. *Communications in Statistics–Theory and Methods*, 22(9):295–306.

Nanda, A. K. and Shaked, M. (2001). The hazard rate and the reversed hazard rate orders, with applications to order statistics. *Annals of the Institute of Statistical Mathematics*, 53(4):853–864.

Nelson, W. (1980). Accelerated life testing-step-stress models and data analyses. *IEEE transactions on reliability*, 29(2):103–108.

Pan, Z. and Balakrishnan*, N. (2010). Multiple-steps step-stress accelerated degradation modeling based on wiener and gamma processes. *Communications in Statistics-Simulation and Computation*, 39(7):1384–1402.

Samanta, D., Ganguly, A., Gupta, A., and Kundu, D. (2019). On classical and bayesian order restricted inference for multiple exponential step stress model. *Statistics*, 53(1):177–195.

Sedyakin, N. (1966). On one physical principle in reliability theory. *Techn. Cybernetics*, 3:80–87.

Shaked, M. and Singpurwalla, N. D. (1983). Inference for step-stress accelerated life tests. *Journal of Statistical Planning and Inference*, 7(4):295–306.

Tang, L.-C. (2003). Multiple-steps step-stress accelerated life test. In *Handbook of Reliability Engineering*, pages 441–455. Springer.

Wang, R. and Fei, H. (2004). Conditions for the coincidence of the tfr, trv and ce models. *Statistical papers*, 45(3):393.

Wang, R.-h. and Fei, H.-l. (2003). Uniqueness of the maximum likelihood estimate of the weibull distribution tampered failure rate model. *Communications in Statistics-Theory and Methods*, 32(12):2321–2338.

Yang, J. and Zhuang, W. (2014). A note on likelihood ratio ordering of order statistics from two samples. *Statistics & Probability Letters*, 84:135–139.