Parametric Analysis of Tampered Random Variable Model for Multiple Step-Stress Life Test

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Abstract

The Tampered Random Variable (TRV) model for multiple step-stress life testing experiment was proposed by Sultana and Dewanji (2020). In this paper, we focus on the parametric inference for this multiple step-stress TRV model based on random right censored lifetime data. The methodology is developed for a general parametric family of distributions. In the special case, we assume the baseline lifetime of the experimental units under normal stress condition to follow exponential distribution with mean θ. We consider both unconstrained and order-restricted maximum likelihood estimation (MLE) of the model parameters. For unconstrained optimization, the closed form solutions of the estimators are derived assuming exponential baseline lifetime distribution. The asymptotic properties of the MLEs are established. Extensive simulation studies are performed to investigate the finite sample properties of the proposed estimators. Finally, the methods are illustrated with the analyses of two real data sets.

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1 Introduction

In many situations, it may be difficult to collect data on failure-time of a product under normal operating conditions due to the continual improvement in manufacturing design and technology. As the product may have a high reliability with substantially long life-spans, sufficient amount of data may not be available during testing under normal conditions. This difficulty is overcome by accelerated life testing (ALT), wherein the units are exposed to higher stress levels than the usual one in order to cause rapid failures. ALT allows the experimenter to apply more severe stresses to obtain information on the model parameters of the lifetime distributions more quickly than would be possible under normal operating conditions. Some key references on ALT model are Nelson (1980), Bhattacharyya and Soejoeti (1989), Madi (1993), Bai and Chung (1992). A special class of the ALT, known as step-stress life testing (SSLT), which allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the experiment. In such a life-testing experiment with k stress levels $s_1, \cdots, s_k$, n identical units are placed on test initially under normal stress level s_1. The stress levels are changed from s_1 to s_2, s_2 to $s_3, \cdots, s_{k-1}$ to s_k, at pre-fixed times $\tau_1, \cdots, \tau_{k-1}$, respectively, known as the tampering times. As in Sultana and Dewanji (2020), let us denote this experiment as k-SSLT. If there are only two stress levels with $k = 2$, then it is known as the simple step-stress life testing, or simple SSLT. The lifetime distribution under the initial normal stress level is termed as the baseline lifetime distribution.

To relate the different lifetime distributions under the different stress levels, there are three different types of modeling assumptions. One of these is the Cumulative Exposure (CE) modeling in which $(k - 1)$ constraints are introduced so that the two distributions at any two successive stress levels coincide at the corresponding change time point to ensure continuity. See, for example, Sedyakin (1966), Nelson (1980) and Tang (2003). Another approach, known as Tampered Failure Rate (TFR) modeling, scales up the failure rates at the successive stress levels. See Bhattacharyya and Soejoeti (1989) and Madi (1993) among others. The third approach, known as Tampered Random Variable (TRV) modeling, scales down the remaining lifetime at the successive stress levels. See Goel (1971) and Sultana and Dewanji (2020) for details. See also Kundu and Ganguly (2017).

There have been some inferential work based on CE and TFR modeling (See Balakrishnan and Xie (2007), Balakrishnan et al. (2009), Pan and Balakrishnan (2010), Mitra et al. (2013), Kateri and Kamps (2015), Samanta et al. (2019), Wang and Fei (2004), Madi (1993), Abdel-Hamid (2009)) in both simple SSLT and k-SSLT experiment. However, for the TRV modeling, the inferential work focused only on the simple SSLT experiment (See DeGroot and Goel (1979); and Bai and Chung (1992); Bai et al. (1993), Abd-Elfattah et al. (2008)). As far as we know, there has not been any inferential work for data from k-SSLT experiments under the TRV modeling approach. In this work, we develop maximum likelihood method to estimate the model parameters for the k-SSLT TRV modeling based on random right censored data.

Following Sultana and Dewanji (2020), the lifetime under the tampered random variable modeling for k-SSLT experiment, denoted by $T_{TRV}^{(k)}$, is given by

$$T_{TRV}^{(k)} = \begin{cases} T, & \text{if } 0 \leq T \leq \tau_1^* \\ \tau_1 + \beta_1(T - \tau_1^*), & \text{if } \tau_1^* < T \leq \tau_2^* \\ \tau_2 + \beta_1\beta_2(T - \tau_2^*), & \text{if } \tau_2^* < T \leq \tau_3^* \\ \vdots & \\ \tau_{k-1} + \prod_{i=1}^{k-1} \beta_i(T - \tau_{k-1}^*), & \text{if } T > \tau_{k-1}^*, \end{cases} \tag{1.1}$$

where $0 < \beta_i \leq 1$ for $i = 1, \cdots, k-1$ are termed as the tampering coefficients, $\tau_1^* = \tau_1$ and $\tau_i^* = \tau_{i-1}^* + \dfrac{(\tau_i - \tau_{i-1})}{\prod_{j=1}^{i-1} \beta_j}$ for $i = 2, \cdots, k-1$. Note that, as $T_{TRV}^{(k)}$ lies between τ_{i-1} and τ_i, the baseline lifetime T takes values between τ_{i-1}^* and τ_i^*, for $i = 1, \ldots, k+1$, with $\tau_0 = \tau_0^* = 0$ and $\tau_{k+1} = \tau_{k+1}^* = \infty$. In this paper, we consider right censored lifetime data from, say, n units under the above k-SSLT experiment with k stress levels and consider estimation of the associated model parameters, including those involved in the distribution of T and the β_i's, using the TRV modeling approach.

It has been noted in many of the previous works that there are some estimability issues in the analysis of data arising from a k-SSLT experiment if there is no failure at some of the stress levels. In particular, if there is no failure at the first stress level under normal condition, the model parameters may not be estimable. In order to circumvent this difficulty, Balakrishnan et al. (2009) have suggested isotonic constraints among the model parameters at different stress levels assuming exponential distribution and CE modeling (See also Samanta et al. (2019)). It is to be noted that, with the tampering coefficients (β_i's) lying between 0 and 1, such isotonic constraints are automatically satisfied in the TRV modeling (1.1), at least for the exponential distribution. In view of the equivalence results of Sultana and Dewanji (2020), the three modeling approaches (namely, CE, TFR, and TRV) are equivalent under exponential distribution. Therefore, the work of the present paper under exponential baseline lifetime distribution reduces to those of Balakrishnan et al. (2009). In this work, we consider general parametric lifetime distributions and also an additional isotonic constraints on the β_i's, as described below.

In Section 2, we consider the issue of identifiability of the model, given by $f_{TRV}^{(k)}$, written as $f_{TRV}^{(k)}(.; \underset{\sim}{\theta}, \underset{\sim}{\beta})$ to indicate its dependence on $\underset{\sim}{\theta}$ and $\underset{\sim}{\beta} = (\beta_1, \cdots, \beta_{k-1})$ explicitly. Section 3 describes the method of maximum likelihood for the model parameters in general, and then obtain analytical solutions for the maximum likelihood estimators (MLEs) assuming exponential distribution for the baseline lifetime, along with some investigation of the asymptotic properties of the MLEs. Since, in practice, the successive stress levels are usually more and more severe in order to ensure rapid failures of the experimental units, it may be reasonable to assume that the tampering coefficients β_i's satisfy the isotonic constraint $1 \geq \beta_1 \geq \cdots \geq \beta_{k-1} > 0$. This represents the likely phenomenon that the increasing stress levels have more impact in tampering (or, shrinking) the effective lifetime. Order-restricted estimation of the model parameters under such isotonic constraint is also considered in Section 3. Section 4 presents some simulation studies to investigate the finite sample properties of the MLEs. We illustrate the proposed methods through the analyses of two real life data sets in Section 5, while Section 6 ends with some concluding remarks.

2 Identifiability

Let $f_T(t;\underset{\sim}{\theta})$ denote the probability density function (PDF) for the baseline lifetime T, where $\underset{\sim}{\theta}$ denotes the associated model parameter(s). Then, for a fixed value of k, the PDF, $f_{TRV}^{(k)}(\cdot)$, of $T_{TRV}^{(k)}$ is given by (See Sultana and Dewanji (2020))

$$f_{TRV}^{(k)}(t;\underset{\sim}{\theta},\underset{\sim}{\beta}) = \begin{cases} f_T(t;\underset{\sim}{\theta}), & 0 \leq t \leq \tau_1 \\ \frac{1}{\beta_1} f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1};\underset{\sim}{\theta}\right), & \tau_1 < t \leq \tau_2 \\ \frac{1}{\beta_1\beta_2} f_T\left(\tau_2^* + \frac{t-\tau_2}{\beta_1\beta_2};\underset{\sim}{\theta}\right), & \tau_2 < t \leq \tau_3 \\ \vdots & \\ \frac{1}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i};\underset{\sim}{\theta}\right), & t > \tau_{k-1}. \end{cases}$$

In order to investigate identifiability of the TRV model given by $f_{TRV}^{(k)}(t;\underset{\sim}{\theta},\underset{\sim}{\beta})$, we consider the equality

$$f_{TRV}^{(k)}(t;\underset{\sim}{\theta}^{(1)},\underset{\sim}{\beta}^{(1)}) = f_{TRV}^{(k)}(t;\underset{\sim}{\theta}^{(2)},\underset{\sim}{\beta}^{(2)}), \quad \text{for all} \quad t > 0, \tag{2.1}$$

with two choices $(\underset{\sim}{\theta}^{(1)},\underset{\sim}{\beta}^{(1)})$ and $(\underset{\sim}{\theta}^{(2)},\underset{\sim}{\beta}^{(2)})$. The model is identifiable if we have $(\underset{\sim}{\theta}^{(1)},\underset{\sim}{\beta}^{(1)}) = (\underset{\sim}{\theta}^{(2)},\underset{\sim}{\beta}^{(2)})$.

Suppose the baseline density $f_T(t;\theta)$ belongs to a standard parametric family of absolutely continuous distributions having identifiability of θ in the usual sense that $f_T(t;\theta^{(1)}) = f_T(t;\theta^{(2)})$ for all $t \geq 0$ implies $\theta^{(1)} = \theta^{(2)}$. Note that, from (2.1), $f_T(t;\underset{\sim}{\theta}^{(1)}) = f_T(t;\underset{\sim}{\theta}^{(2)})$ for $0 \leq t \leq \tau_1(> 0)$. Since this is true for all τ_1, letting

$\tau_1 \to \infty$, we have $\underset{\sim}{\theta}^{(1)} = \underset{\sim}{\theta}^{(2)} = \underset{\sim}{\theta}$, say. Next, using (2.1) for $\tau_1 < t \le \tau_2$, we have

$$
\begin{aligned}
&\frac{1}{\beta_1^{(1)}} f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(1)}}; \underset{\sim}{\theta}\right) = \frac{1}{\beta_1^{(2)}} f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(2)}}; \underset{\sim}{\theta}\right). \\
\iff & \frac{f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(1)}}; \underset{\sim}{\theta}\right)}{f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(2)}}; \underset{\sim}{\theta}\right)} = \frac{\beta_1^{(1)}}{\beta_1^{(2)}} = c, \quad \text{say.} \\
\iff & \beta_1^{(1)} = c\, \beta_1^{(2)} \quad \text{and} \quad f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(1)}}; \underset{\sim}{\theta}\right) = c\, f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(2)}}; \underset{\sim}{\theta}\right).
\end{aligned}
$$

This holds for all $\tau_1 < t \le \tau_2$. Therefore, integrating over $(\tau_1, \tau_2]$ with respect to t, we get

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(1)}}; \underset{\sim}{\theta}\right) \mathrm{d}t = c \int_{\tau_1}^{\tau_2} f_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1^{(2)}}; \underset{\sim}{\theta}\right) \mathrm{d}t \\
\Rightarrow\ & c\, \beta_1^{(2)} \int_{\tau_1^*}^{\tau_1^* + \frac{\tau_2-\tau_1}{c\ \beta_1^{(2)}}} f_T(u; \underset{\sim}{\theta}) \mathrm{d}u = c\, \beta_1^{(2)} \int_{\tau_1^*}^{\tau_1^* + \frac{\tau_2-\tau_1}{\beta_1^{(2)}}} f_T(u; \underset{\sim}{\theta}) \mathrm{d}u. \\
\Rightarrow\ & F_T\left(\tau_1^* + \frac{\tau_2-\tau_1}{c\ \beta_1^{(2)}}; \underset{\sim}{\theta}\right) - F_T\left(\tau_1^*; \underset{\sim}{\theta}\right) = F_T\left(\tau_1^* + \frac{\tau_2-\tau_1}{\beta_1^{(2)}}; \underset{\sim}{\theta}\right) - F_T\left(\tau_1^*; \underset{\sim}{\theta}\right). \\
\Rightarrow\ & F_T\left(\tau_1^* + \frac{\tau_2-\tau_1}{c\ \beta_1^{(2)}}; \underset{\sim}{\theta}\right) = F_T\left(\tau_1^* + \frac{\tau_2-\tau_1}{\beta_1^{(2)}}; \underset{\sim}{\theta}\right).
\end{aligned}
$$

Assuming that $f_T(x; \underset{\sim}{\theta}) > 0$, for all $x > 0$, the above equality implies

$$
\begin{aligned}
&\tau_1^* + \frac{\tau_2-\tau_1}{c\ \beta_1^{(2)}} = \tau_1^* + \frac{\tau_2-\tau_1}{\beta_1^{(2)}} \\
&\Rightarrow c = 1, \qquad \text{which implies} \quad \beta_1^{(1)} = \beta_1^{(2)}.
\end{aligned}
$$

While proceeding in the similar manner considering the equality (2.1) for the successive time intervals defined by the tampering times τ_i's, note that τ_i^* , for $i = 2, \cdots, k-1$, depends on $\beta_j,\ j = 1, \cdots, i-1$. This, however, does not create a problem since the previous β_j's are proved to be equal prior to considering τ_i^* for the interval $(\tau_i, \tau_{i+1}]$. We thus obtain

$$\beta_j^{(1)} = \beta_j^{(2)}, \quad \text{for} \quad j = 2, \cdots, (k-1).$$

Hence, we have $(\underset{\sim}{\theta}^{(1)}, \underset{\sim}{\beta}^{(1)}) = (\underset{\sim}{\theta}^{(2)}, \underset{\sim}{\beta}^{(2)})$ and, thus, the model is identifiable. The above proof of identifiability indicates some problem in estimating the model parameters and the tampering coefficients unless there are failures at each of k stress levels including the normal one. See the next section for such results and the discussion thereof under the general parametric model for the baseline lifetime distribution and, in particular, the exponential baseline lifetime distribution.

3 Maximum Likelihood Estimation

Suppose the censoring random variable is denoted by C. A typical observation consists of the variable $\Delta = \mathbf{I}(T_{TRV}^{(k)} < C)$ indicating the failure of the unit before the censoring time and $X = min(T_{TRV}^{(k)}, C)$ giving the time of observation. Then the observed data is $\{(x_i, \delta_i);\ i = 1, \cdots, n\}$ and the likelihood function can be written as

$$L(\underset{\sim}{\theta}, \underset{\sim}{\beta}) \propto \prod_{i=1}^{n} \left[f_{TRV}^{(k)}(x_i; \underset{\sim}{\theta}, \underset{\sim}{\beta})^{\delta_i} \left(\bar{F}_{TRV}^{(k)}(x_i; \underset{\sim}{\theta}, \underset{\sim}{\beta}) \right)^{1-\delta_i} \right], \tag{3.1}$$

where $\bar{F}_{TRV}^{(k)}(\cdot; \underset{\sim}{\theta}, \underset{\sim}{\beta}) = 1 - F_{TRV}^{(k)}(\cdot; \underset{\sim}{\theta}, \underset{\sim}{\beta})$. Let us define the intervals $I_1 = [0, \tau_1]$, $I_i = (\tau_{i-1}, \tau_i]$, for $i = 2, \cdots, (k-1)$ and $I_k = (\tau_{k-1}, \infty)$. Define D_j as the set of individuals failing in I_j and C_j as the set of individuals censored in I_j, for $j = 1, \cdots, k$. Write $d_j = |D_j|$ and $c_j = |C_j|$, for $j = 1, \cdots, k$ and $d = \sum_{j=1}^{k} d_j \leq n$. The likelihood function (3.1) can be rewritten as

$$L(\underset{\sim}{\theta}, \underset{\sim}{\beta}) \propto \prod_{i \in D_1} f_T(x_i; \underset{\sim}{\theta}) \times \prod_{i \in C_1} \bar{F}_T(x_i; \underset{\sim}{\theta}) \times \prod_{j=2}^{k} \Bigg[\Bigg\{ \prod_{i \in D_j} \frac{1}{\prod_{l=1}^{j-1} \beta_l} f_T(\tau_{j-1}^* + \frac{x_i - \tau_{j-1}}{\prod_{l=1}^{j-1} \beta_l}; \underset{\sim}{\theta}) \Bigg\} \times \prod_{i \in C_j} \bar{F}_T(\tau_{j-1}^* + \frac{x_i - \tau_{j-1}}{\prod_{l=1}^{j-1} \beta_l}; \underset{\sim}{\theta}) \Bigg]. \tag{3.2}$$

Note that, for $k = 2$, this likelihood reduces to that of Abd-Elfattah et al. (2008) for a simple SSLT model. As indicated in the proof of the identifiability result, it is clear from (3.2) that, if $d_1 = 0$, the parameter θ may be confounded with the β_j's and hence may not be estimable. Further, if $d_j = 0$ for some other j, the corresponding β_{j-1} may be estimated at some limiting value. See the discussion with exponential baseline lifetime model below followed by the same with general parametric model. Nevertheless, even if all the d_j's are positive (in case of a large enough sample), it is clear that no closed form analytic solution for the estimates of the model parameters exists in general. As a result, maximum likelihood estimation of $(\underset{\sim}{\theta}, \underset{\sim}{\beta})$ is carried out using some numerical maximization method. For example, with a general baseline lifetime distribution, we suggest the **optim** function in R software to obtain the maximum likelihood estimates of the model parameters. However, for exponential baseline lifetime distribution, it is possible to find closed form solutions of the estimators as discussed in the following.

Exponential Baseline Lifetime Distribution

Let us consider the following reparametrization of the parameters β_i's:

$$\gamma_j = \prod_{l=1}^{j} \beta_l, \quad \text{for } j = 1, \cdots, (k-1).$$

Clearly, $1 \geq \gamma_1 \geq \cdots \geq \gamma_{k-1} > 0$, since $0 < \beta_j \leq 1$, for all $j = 1, \cdots, (k-1)$. Therefore, the likelihood function in terms of the parameters θ and $\gamma_1, \cdots, \gamma_{k-1}$

under the assumption of exponential baseline lifetime distribution is

$$L(\theta, \underset{\sim}{\gamma}) \propto \prod_{i \in D_1} \left\{ \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \right\} \prod_{i \in C_1} \left\{ e^{-\frac{x_i}{\theta}} \right\} \prod_{j=2}^{k} \Bigg\{ \prod_{i \in D_j} \frac{1}{\theta\, \gamma_{j-1}} e^{-\frac{\tau_{j-1}^* + \frac{x_i - \tau_{j-1}}{\gamma_{j-1}}}{\theta}} \times \prod_{i \in C_j} e^{-\frac{\tau_{j-1}^* + \frac{x_i - \tau_{j-1}}{\gamma_{j-1}}}{\theta}} \Bigg\},$$

where $\underset{\sim}{\gamma} = (\gamma_1, \cdots, \gamma_{k-1})$. The log-likelihood function is given by

$$\begin{aligned} l(\theta, \underset{\sim}{\gamma}) \propto -d \log \theta - \sum_{j=1}^{k-1} d_{j+1} \log \gamma_j - \frac{1}{\theta} \Bigg[& \sum_{i \in D_1 \cup C_1} x_i + \sum_{i \in D_2 \cup C_2} \left\{ \tau_1 + \frac{x_i - \tau_1}{\gamma_1} \right\} \\ & + \sum_{i \in D_3 \cup C_3} \left\{ \tau_1 + \frac{\tau_2 - \tau_1}{\gamma_1} + \frac{x_i - \tau_2}{\gamma_2} \right\} + \cdots \\ & + \sum_{i \in D_k \cup C_k} \left\{ \tau_1 + \frac{\tau_2 - \tau_1}{\gamma_1} + \cdots + \frac{\tau_{k-1} - \tau_{k-2}}{\gamma_{k-2}} + \frac{x_i - \tau_{k-1}}{\gamma_{k-1}} \right\} \Bigg]. \end{aligned}$$

$$\begin{aligned} = -d \log \theta - \sum_{j=1}^{k-1} d_{j+1} \log \gamma_j - \frac{1}{\theta} \Bigg[& \sum_{i \in D_1 \cup C_1} x_i + \tau_1 \sum_{j=2}^{k} (d_j + c_j) \\ & + \frac{1}{\gamma_1} \left\{ \sum_{i \in D_2 \cup C_2} (x_i - \tau_1) + (\tau_2 - \tau_1) \sum_{j=3}^{k} (d_j + c_j) \right\} \\ & + \frac{1}{\gamma_{k-2}} \left\{ \sum_{i \in D_{k-1} \cup C_{k-1}} (x_i - \tau_{k-2}) + (\tau_{k-1} - \tau_{k-2})(d_k + c_k) \right\} \\ & + \frac{1}{\gamma_{k-1}} \sum_{i \in D_k \cup C_k} (x_i - \tau_{k-1}) \Bigg]. \end{aligned}$$

$$= -d\log\theta - \sum_{j=1}^{k-1} d_{j+1}\log\gamma_j - \frac{1}{\theta}\left[A_1 + \frac{A_2}{\gamma_1} + \cdots + \frac{A_k}{\gamma_{k-1}}\right], \tag{3.3}$$

say, where, $A_1 = \sum_{i\in D_1\cup C_1} x_i + \tau_1 \sum_{j=2}^{k}(d_j + c_j)$, $A_j = \sum_{i\in D_j\cup C_j}(x_i - \tau_{j-1}) + (\tau_j - \tau_{j-1})\sum_{l=j+1}^{k}(d_l + c_l)$, for $j = 2, \cdots, (k-1)$ and $A_k = \sum_{i\in D_k\cup C_k}(x_i - \tau_{k-1})$. Note that each A_j represents the 'total time on test' in I_j, for $j = 1, \cdots, k$. Let us again consider the re-parametrization

$$\alpha_1 = \frac{1}{\theta},\ \alpha_j = \frac{\alpha_1}{\gamma_{j-1}},\ j = 2, \cdots, k.$$

Clearly, the new parameters α_j's satisfy the isotonic constraints $0 < \alpha_1 \leq \cdots \leq \alpha_k < \infty$, since $1 \geq \gamma_1 \geq \cdots \geq \gamma_{k-1} > 0$. The log-likelihood function (3.3) in terms of α_j's is given by

$$l(\underset{\sim}{\alpha}) \propto \sum_{j=1}^{k} d_j \log\alpha_j - \sum_{j=1}^{k} A_j\alpha_j, \tag{3.4}$$

where $\underset{\sim}{\alpha}$ is the vector of α_j's. Note that, from this expression of the log-likelihood function, unique MLEs of the α_j's in the unconstrained space exist when $d_j > 0$ for all $j = 1, \cdots, k$. Assuming the same and maximizing the log-likelihood function $l(\underset{\sim}{\alpha})$, we get the unconstrained MLEs of the α_j's as d_j/A_j, for $j = 1, \cdots, k$, respectively. Therefore, the constrained MLEs of α_j's, denoted by $\hat{\alpha}_j$'s, are obtained by using the Pool Adjacent Violators Algorithm (PAVA) on the d_j/A_j's. That is, $\hat{\alpha}_j$ is given by

$$\hat{\alpha}_j = \max_{s\leq j}\min_{t\geq j}\alpha^{s,t},$$

$$\text{where } \alpha^{s,t} = \arg\max_{\alpha}\sum_{i=s}^{t}\Big[d_i\log\alpha - A_i\alpha\Big] = \frac{\sum_{i=s}^{t} d_i}{\sum_{i=s}^{t} A_i}.$$

See (2.8) in Barlow and Brunk (1972, p141). This holds for all $j = 1, \cdots, k$. The MLEs of the original parameters θ and $\underset{\sim}{\beta}$ are obtained, using the invariance property of the MLE, as $\hat{\theta} = 1/\hat{\alpha}_1$ and $\hat{\beta}_j = \hat{\alpha}_j/\hat{\alpha}_{j+1}$, for $j = 1, \cdots, (k-1)$. Clearly, we have

$0 < \hat{\beta}_j \leq 1$ for all j because of the isotonic constraint on the $\hat{\alpha}_j$'s.

Note that, because of the application of PAVA to obtain the MLEs in the constrained space, some parameters may be estimated at the corresponding limits when some of the d_j's are zero (See some examples in the following), which ensures estimability of some of the parameters, or some parametric functions. For example, if $d_1 = 0$ and $d_j > 0$ for $j = 2, \cdots, k$, then MLE of θ does not exist (since $\hat{\alpha}_1 = 0$), but MLEs of α_j, for $j = 2, \cdots, k$, exist which in turn give the MLEs of $\theta\beta_1, \beta_2, \cdots, \beta_{k-1}$. Note that the probability of such an event (that is, $\{d_1 = 0\}$) is $[\bar{F}_T(\tau_1; \theta)]^n$, which tends to 0 as $n \to \infty$. Similarly, if some of the other d_j's are zero, then the PAVA will be able to deal with that to give meaningful estimates as long as some of the previous d_j's are positive.

Suppose $d_j = 0$, for some $2 \leq j \leq k$. From the likelihood function (3.4), it is clear that it is a decreasing function of α_j and hence, is maximum when α_j takes the minimum possible value. Since the α_j's satisfy the monotonic constraint $0 < \alpha_1 \leq \cdots \leq \alpha_k$, the likelihood maximizes at $\alpha_j = \alpha_{j-1}$, for $j = 2, \cdots, k$, which gives $\hat{\beta}_{j-1} = 1$ (See first row of Table 8). In another extreme case when $d_k = c_k = 0$, α_k does not appear in the likelihood; therefore, the parameter β_{k-1} is not estimable.

General Parametric Baseline Lifetime Distribution

Now consider a general parametric baseline lifetime distribution giving the likelihood function (3.2). As argued before, one cannot estimate the parameter θ if $d_1 = 0$. So we consider the case $d_1 > 0$, but some of the d_j's, for $j = 2, \cdots, k$, may be zero. By reparametrization of the parameters β_i's into γ_i's, as in the case of exponential distribution, the likelihood function (3.2) can be rewritten as

$$L(\underset{\sim}{\theta}, \underset{\sim}{\gamma}) = \prod_{i \in D_1} f_T(x_i; \underset{\sim}{\theta}) \times \prod_{i \in C_1} \bar{F}_T(x_i; \underset{\sim}{\theta}) \times \prod_{j=2}^{k} \Bigg[\Bigg\{ \prod_{i \in D_j} \frac{1}{\gamma_{j-1}} f_T(\tau^*_{j-1} + \frac{x_i - \tau_{j-1}}{\gamma_{j-1}}; \underset{\sim}{\theta}) \Bigg\} \times \prod_{i \in C_j} \bar{F}_T(\tau^*_{j-1} + \frac{x_i - \tau_{j-1}}{\gamma_{j-1}}; \underset{\sim}{\theta}) \Bigg]. \tag{3.5}$$

Suppose now $d_j = 0$ for some $j = 2, \cdots, k$. Then, it is clear that γ_{j-1} appears only for the contribution corresponding to those in C_j, given by $\prod_{i \in C_j} \bar{F}_T(\tau^*_{j-1} + \dfrac{x_i - \tau_{j-1}}{\gamma_{j-1}}; \underset{\sim}{\theta})$ in the likelihood function (3.5). The likelihood function (3.5), therefore, becomes an increasing function of γ_{j-1} (since survival function is a non-increasing function) and, hence, the likelihood is maximum when γ_{j-1} takes the maximum possible value. Since γ_i's satisfy the monotonic constraint $1 \geq \gamma_1 \geq \cdots \geq \gamma_{k-1} > 0$, the likelihood maximizes at $\gamma_{j-1} = \gamma_{j-2}$, if $j = 3, \cdots, k$, and at $\gamma_{j-1} = 1$, if $j = 2$, which gives $\beta_{j-1} = 1$.

Thus, when some of the d_j's are 0, there is some dimension reduction while obtaining the MLEs. Define $\underset{\sim}{\psi}$ as the vector of estimable model parameters. Let $\underset{\sim}{\psi_0}$ denote the true value of $\underset{\sim}{\psi}$ with $\underset{\sim}{\hat{\psi}}$ being the corresponding MLE. Since the baseline lifetime distribution given by the density $f(\cdot;\theta)$ is assumed to satisfy the standard regularity conditions, the distribution given by $f^{(k)}_{TRV}(t;\theta,\beta)$ also satisfies those conditions of differentiability and continuity. Assuming further that the corresponding Fisher information matrix is positive definite, we have the following asymptotic results using Theorem 5.1 of (Lehmann and Casella, 1998, p463-465).

Result: Following the standard parametric analysis, under some regularity conditions as indicated above, we have

(i) $\underset{\sim}{\hat{\psi}} \xrightarrow{P} \underset{\sim}{\psi_0}$ and

(ii) $\sqrt{n}(\underset{\sim}{\hat{\psi}} - \underset{\sim}{\psi_0})$ is asymptotically a mean zero normal random vector with variance covariance matrix estimated by the inverse of the corresponding hessian matrix computed at the MLE.

Constrained Estimation

The isotonic constraint $1 \geq \beta_1 \geq \cdots \geq \beta_{k-1} > 0$, as discussed in Section 1, is naturally of interest. Note that the isotonic constraints considered by Balakrishnan et al. (2009) and Samanta et al. (2017, 2019) under the CE model are already satisfied by the assumption $0 < \beta_i \leq 1$ for all i under the TRV model. Therefore, the constraint $1 \geq \beta_1 \geq \cdots \geq \beta_{k-1} > 0$ is an additional one which has not been considered before. In this section, we consider estimation of the model parameters under this isotonic constraint in the framework of k-SSLT with TRV modeling. Let us consider the re-parametrization from $(\beta_1, \cdots, \beta_{k-1})$ to $(\rho_1, \cdots, \rho_{k-1})$ as given below.

$$\beta_1 = \rho_1,\ \beta_2 = \rho_1\rho_2,\ \cdots, \beta_{k-1} = \prod_{i=1}^{k-1} \rho_i, \quad \text{so that}$$

$$\rho_1 = \beta_1,\ \rho_2 = \frac{\beta_2}{\beta_1},\ \cdots, \rho_{k-1} = \frac{\beta_{k-1}}{\beta_{k-2}}.$$

This gives

$$\gamma_1 = \beta_1 = \rho_1,\ \gamma_2 = \beta_1\beta_2 = \rho_1^2\rho_2, \cdots, \gamma_{k-1} = \prod_{i=1}^{k-1} \beta_i = \rho_{k-1}\rho_{k-2}^2 \cdots \rho_1^{k-1}.$$

Note that, since $0 < \beta_i \leq 1$, for all i, and $1 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{k-1} > 0$, so the ρ_i's satisfy the constraints $0 < \rho_i \leq 1$, for all $i = 1, \cdots, (k-1)$. Thus, this re-parametrization of β_i's into ρ_i's reduces the constrained optimization problem into a simpler one without any isotonic constraint. Rewriting the likelihood (3.2) in terms of θ and the new set of parameters $\underset{\sim}{\rho} = (\rho_1, \cdots, \rho_{k-1})$, one can easily use any numerical maximization method to obtain the corresponding MLEs. In our paper, we have used the **optim** function in R software for this purpose.

Assuming first exponential baseline lifetime distribution for simplicity, the log-likelihood function (3.3) in terms of the parameters θ and $\rho_1, \cdots, \rho_{k-1}$ simplifies to

$$L(\theta, \underset{\sim}{\rho}) \propto -d \log\theta - \sum_{j=1}^{k-1} d_{j+1} \left[\sum_{l=1}^{j} (j-l+1) \log\rho_l \right] - \frac{1}{\theta}\left[A_1 + \frac{A_2}{\rho_1} + \cdots \frac{A_k}{\prod_{l=1}^{j} \rho_l^{j-l+1}} \right],$$

where A_j's are the same as those defined earlier in this section. If $d_j > 0$, for all $j = 1, \cdots, k$, the MLEs are obtained using numerical maximization method, as discussed previously. Otherwise, assuming $d_1 > 0$, let us consider the case when $d_j = 0$, for some $2 \leq j \leq (k-1)$. Noting that the log-likelihood function (3.3) is an increasing function of γ_{j-1} for $d_j = 0$, the likelihood is maximum when γ_{j-1} takes the maximum possible value, that is $\gamma_{j-1} = \gamma_{j-2}$, if $3 \leq j \leq (k-1)$, and $\gamma_{j-1} = 1$, if $j = 2$. This gives $\beta_{j-1} = 1$, as argued before. Since the β_j's satisfy the monotonic constraint $1 \geq \beta_1 \geq \cdots \beta_{k-1} > 0$, we have $\beta_1 = \cdots = \beta_{j-1} = 1$ and the dimension of the parameters to be estimated is reduced to $(k - j + 1)$. Extending this argument, it can be concluded that, if j_0 is the largest j such that $d_j = 0$, then we can restrict to the reduced parameter space with $\beta_1 = \cdots = \beta_{j_0-1} = 1$.

The same argument follows with the general parametric baseline lifetime distribution and the likelihood function (3.5). That is, if j_0 is the largest j such that $d_j = 0$, then we can restrict to the reduced parameter space with $\beta_1 = \cdots = \beta_{j_0-1} = 1$. (See Table 9 in Section 5). This essentially translates into $\rho_1 = \cdots = \rho_{j_0-1} = 1$.

4 Simulation Studies

We conduct several simulation studies to investigate the finite sample properties of the maximum likelihood estimators discussed in Section 3. In order to simulate a data set of size n from the TRV model of Section 2, we first simulate n observations from the assumed baseline lifetime distribution $F(\cdot; \underset{\sim}{\theta})$ for the random variable T along with the corresponding censoring times, simulated from the distribution $G(\cdot)$ of C. For a given k-SSLT with prefixed tampering time points $\tau_1 < \cdots < \tau_{k-1}$ and tampering coefficients $\beta_1, \cdots, \beta_{k-1}$, one simulated lifetime $T_{TRV}^{(k)}$ is obtained from a simulated T value by using (1.1). Then, the observed variable (X, Δ) is obtained as $X = \min(T_{TRV}^{(k)}, C)$ and $\Delta = \mathcal{I}(T_{TRV}^{(k)} \leq C)$. Thus, we have a simulated data set of the form (x_i, δ_i), for $i = 1, \cdots, n$. This simulation study is carried out $10,000$ times to get $10,000$ such simulated data sets.

For each simulated data set, we obtain the maximum likelihood estimates of the model parameters $(\underset{\sim}{\theta}, \underset{\sim}{\beta})$, along with the corresponding standard errors, using the method discussed in Section 3. We also obtain an asymptotic 95% confidence interval for each parameter based on the normal approximation. Then, we compute and report the average of these $10,000$ estimates , denoted by AE, and average of the corresponding standard errors, denoted by ASE, for each parameter. We also report the sample standard errors, denoted by SSE, as the standard deviation of the $10,000$ estimates. Cover percentage of an estimator is estimated by the proportion of times the corresponding 95% asymptotic confidence intervals contain the true value of the parameter and is denoted by CP. Three different choices of the sample size $n = 50, 150$, and 250 are taken to study the behavior of the estimates with change in the sample size. Note that, the probability that there is no failure at the jth stress (that is, $d_j = 0$) is $\{1 - P[\tau_{j-1} < T_{TRV}^{(k)} \leq \tau_j]\}^n$, for $j = 1, \cdots, (k-1)$ and $\tau_0 = 0$, which tends to 0 as $n \to \infty$. Since, in our simulation studies, we have taken moderate to large values of n, the probability of no failure in any stress level is rather small. However, if any simulated data has such zero frequency at any stress level, we discard that data and consider the remaining ones for estimation.

First we consider a simple SSLT (that is, $k = 2$). Two different choices of $\tau_1 = 0.35, 0.5$ and $\beta_1 = 0.8, 0.5$ are taken. Considering exponential baseline lifetime distribution with mean 1, closed form solutions of the estimators are obtained, as discussed in Section 3. We have also considered the Weibull baseline lifetime distribution with two choices of the shape parameter $\alpha = 0.8$ (decreasing failure rate) and 1.2 (increasing failure rate) and the common scale parameter 1. The censoring distribution in all the simulation studies is taken to be exponential with mean μ. The failure probability is defined as $P[T_{TRV}^{(k)} \leq C] = \frac{1}{\mu}\int_0^\infty \left[\int_0^c f_{TRV}^{(k)}(t; \underset{\sim}{\theta}, \underset{\sim}{\beta})\mathrm{d}t\right] e^{-\frac{c}{\mu}}\mathrm{d}c = p_f$, say. Different choices of p_f are taken as 0.6 and 0.8 in order to study their impact on the estimates. Thus, for a given choice of $(\underset{\sim}{\beta}, \underset{\sim}{\theta})$, the value of μ is determined from $p_f = 0.6$ (or 0.8). The results are presented in Tables 1-3 for exponential, Weibull (DFR) and Weibull (IFR) baseline lifetime distributions, respectively.

Next we consider a 3-SSLT with two different choices of (τ_1, τ_2) as $(0.3, 0.6)$ and $(0.3, 0.8)$, and (β_1, β_2) as $(0.8, 0.6)$ and $(0.8, 0.5)$. We consider the same exponential and Weibull baseline lifetime distributions as for the simple SSLT above. The censoring distribution is taken as exponential with mean μ, chosen to give values of p_f as 0.6 and 0.8, as discussed above. The results are presented in Tables 4-6 for exponential, Weibull (DFR) and Weibull (IFR) baseline lifetime distributions, respectively.

Finally, we consider the constrained estimation as discussed at the end of Section

3. We consider a 3-SSLT with choice of (τ_1, τ_2) as $(0.3, 0.6)$ and (β_1, β_2) as $(0.8, 0.5)$. The baseline lifetime distribution is taken to be exponential with mean rate 1. Similar to the above simulation studies, the censoring distribution is taken as exponential with mean μ, chosen to give values of p_f as 0.6 and 0.8. The results are presented in Table 7.

As expected, the estimates are closer to the true values with the increase in sample size. It is also observed that the corresponding ASE and SSE decrease and are closer to each other as the sample size increases. Also, the CP values converge to 0.95 with increase in sample size providing some evidence in favour of asymptotic normality. As p_f increases, the standard errors of the estimates (reported by ASE and SSE) decrease, as expected, since an increase in p_f results in more information on the failure process. Also, as τ_1 increases, there is more information on θ (for exponential distribution) and $(\theta,\ \alpha)$ (for Weibull distribution) and hence, the standard errors of the corresponding estimates decrease. Similarly, with an increase in the difference between τ_1 and τ_2 in 3-SSLT, there is more information on β_1, resulting in a decrease of the standard error of the corresponding estimate (See Tables 4-6).

Table 1: Simulation results for the exponential baseline lifetime distribution with scale parameter $\theta = 1$, and different choices of τ_1 and β_1 under simple SSLT.

				$p_f = 0.6$				$p_f = 0.8$			
τ_1	β_1	n	Parameter	AE	ASE	SSE	CP	AE	ASE	SSE	CP
0.35	0.5	50	θ	1.095	0.330	0.370	0.936	1.088	0.312	0.354	0.939
			β_1	0.503	0.187	0.188	0.940	0.500	0.166	0.170	0.943
		150	θ	1.018	0.167	0.175	0.949	1.019	0.160	0.166	0.950
			β_1	0.511	0.109	0.111	0.942	0.504	0.096	0.100	0.939
		250	θ	1.018	0.129	0.132	0.955	1.018	0.123	0.127	0.947
			β_1	0.502	0.083	0.084	0.938	0.499	0.074	0.075	0.941
0.35	0.8	50	θ	1.088	0.318	0.320	0.967	1.081	0.303	0.306	0.970

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Table 1 – continued from previous page

τ_1	β_1	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
			β_1	0.760	0.275	0.200	0.935	0.765	0.250	0.186	0.947
		150	θ	1.028	0.166	0.156	0.981	1.024	0.160	0.149	0.980
			β_1	0.793	0.168	0.140	0.956	0.795	0.151	0.133	0.950
		250	θ	1.012	0.126	0.123	0.961	1.011	0.121	0.117	0.963
			β_1	0.807	0.132	0.120	0.959	0.806	0.119	0.110	0.966
0.5	0.5	50	θ	1.034	0.265	0.273	0.942	1.028	0.248	0.255	0.946
			β_1	0.526	0.197	0.191	0.952	0.526	0.166	0.168	0.948
		150	θ	1.025	0.149	0.149	0.952	1.023	0.140	0.144	0.954
			β_1	0.501	0.106	0.107	0.947	0.499	0.091	0.097	0.951
		250	θ	1.008	0.112	0.111	1.005	0.949	0.105	0.103	0.954
			β_1	0.505	0.083	0.083	0.943	0.503	0.071	0.070	0.952
0.5	0.8	50	θ	1.074	0.274	0.273	0.967	1.066	0.257	0.254	0.969
			β_1	0.764	0.277	0.202	0.948	0.767	0.241	0.187	0.946
		150	θ	1.022	0.146	0.140	0.962	1.016	0.137	0.131	0.977
			β_1	0.794	0.168	0.139	0.954	0.796	0.145	0.132	0.949
		250	θ	1.010	0.111	0.112	0.947	1.006	0.105	0.104	0.948
			β_1	0.800	0.131	0.119	0.945	0.801	0.113	0.105	0.958

Table 2: Simulation results for the Weibull baseline lifetime distribution with shape parameter $\alpha = 0.8$ and $\theta = 1$, and different choices of τ_1 and β_1 under simple SSLT.

				$p_f = 0.6$				$p_f = 0.8$			
τ_1	β_1	n	Parameter	AE	ASE	SSE	CP	AE	ASE	SSE	CP
0.35	0.5	50	α	0.856	0.180	0.183	0.964	0.854	0.162	0.168	0.958
			θ	1.094	0.498	0.523	0.941	1.069	0.427	0.441	0.943
			β_1	0.584	0.335	0.266	0.937	0.576	0.302	0.250	0.942
		150	α	0.819	0.099	0.101	0.960	0.815	0.088	0.089	0.945
			θ	1.035	0.256	0.266	0.943	1.027	0.229	0.245	0.942
			β_1	0.536	0.182	0.187	0.942	0.531	0.165	0.171	0.946
		250	α	0.807	0.075	0.077	0.948	0.810	0.067	0.071	0.946
			θ	1.035	0.197	0.205	0.950	1.019	0.174	0.179	0.949
			β_1	0.513	0.136	0.136	0.942	0.518	0.125	0.129	0.944
0.35	0.8	50	α	0.808	0.168	0.144	0.968	0.809	0.149	0.131	0.974
			θ	1.133	0.517	0.537	0.966	1.106	0.444	0.420	0.962
			β_1	0.775	0.434	0.243	0.942	0.776	0.400	0.236	0.946
		150	α	0.804	0.097	0.085	0.966	0.805	0.087	0.071	0.978
			θ	1.055	0.259	0.242	0.966	1.044	0.231	0.212	0.972
			β_1	0.791	0.264	0.184	0.950	0.796	0.245	0.178	0.954
		250	α	0.800	0.074	0.067	0.964	0.803	0.067	0.061	0.966
			θ	1.034	0.193	0.179	0.974	1.027	0.174	0.162	0.976
			β_1	0.798	0.207	0.160	0.958	0.800	0.191	0.154	0.952
0.5	0.5	50	α	0.847	0.167	0.167	0.960	0.840	0.151	0.153	0.948
			θ	1.063	0.411	0.460	0.940	1.051	0.359	0.457	0.945

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Table 2 – continued from previous page

τ_1	β_1	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
			β_1	0.583	0.322	0.262	0.943	0.560	0.277	0.237	0.944
		150	α	0.815	0.092	0.087	0.974	0.815	0.084	0.081	0.964
			θ	1.014	0.210	0.202	0.942	1.008	0.188	0.187	0.944
			β_1	0.527	0.173	0.161	0.952	0.531	0.156	0.148	0.964
		250	α	0.806	0.071	0.074	0.944	0.804	0.064	0.066	0.954
			θ	1.013	0.162	0.167	0.942	1.012	0.147	0.154	0.943
			β_1	0.518	0.132	0.141	0.948	0.511	0.117	0.123	0.943
0.5	0.8	50	α	0.814	0.161	0.153	0.960	0.817	0.148	0.137	0.954
			θ	1.141	0.444	0.431	0.974	1.116	0.385	0.468	0.970
			β_1	0.760	0.414	0.247	0.943	0.761	0.373	0.237	0.947
		150	α	0.808	0.091	0.082	0.972	0.809	0.084	0.071	0.978
			θ	1.027	0.209	0.190	0.968	1.023	0.190	0.175	0.975
			β_1	0.809	0.258	0.175	0.970	0.816	0.236	0.168	0.972
		250	α	0.797	0.070	0.065	0.952	0.800	0.064	0.059	0.960
			θ	1.034	0.165	0.156	0.964	1.022	0.148	0.140	0.966
			β_1	0.793	0.200	0.157	0.956	0.799	0.182	0.144	0.954

Table 3: Simulation results for the Weibull baseline lifetime distribution with shape parameter $\alpha = 1.2$ and $\theta = 1$, and different choices of τ_1 and β_1 under simple SSLT.

				$p_f = 0.6$				$p_f = 0.8$			
τ_1	β_1	n	Parameter	AE	ASE	SSE	CP	AE	ASE	SSE	CP
0.35	0.5	50	α	1.271	0.306	0.307	0.960	1.279	0.276	0.266	0.978
			θ	1.136	0.513	0.543	0.943	1.069	0.411	0.422	0.942
			β_1	0.562	0.320	0.273	0.944	0.568	0.297	0.254	0.946
		150	α	1.242	0.170	0.178	0.946	1.235	0.152	0.160	0.940
			θ	1.011	0.230	0.225	0.947	1.008	0.207	0.204	0.948
			β_1	0.539	0.181	0.170	0.944	0.535	0.165	0.156	0.941
		250	α	1.221	0.129	0.127	0.958	1.216	0.115	0.114	0.960
			θ	1.011	0.177	0.180	0.944	1.010	0.161	0.163	0.940
			β_1	0.520	0.137	0.134	0.956	0.517	0.125	0.123	0.958
0.35	0.8	50	α	1.225	0.289	0.229	0.958	1.230	0.258	0.205	0.968
			θ	1.142	0.494	0.480	0.980	1.100	0.412	0.361	0.978
			β_1	0.770	0.431	0.253	0.945	0.778	0.400	0.239	0.946
		150	α	1.215	0.166	0.142	0.972	1.206	0.147	0.126	0.972
			θ	1.038	0.235	0.214	0.984	1.044	0.217	0.195	0.986
			β_1	0.798	0.265	0.185	0.946	0.794	0.244	0.182	0.948
		250	α	1.199	0.126	0.114	0.956	1.202	0.113	0.104	0.960
			θ	1.037	0.182	0.180	0.988	1.027	0.163	0.157	0.984
			β_1	0.794	0.206	0.170	0.938	0.800	0.192	0.159	0.950
0.5	0.5	50	α	1.277	0.282	0.273	0.966	1.271	0.258	0.252	0.970
			θ	1.039	0.334	0.330	0.946	1.028	0.293	0.290	0.944

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Table 3 – continued from previous page

τ_1	β_1	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
			β_1	0.585	0.301	0.250	0.946	0.573	0.268	0.243	0.942
		150	α	1.233	0.157	0.152	0.968	1.229	0.144	0.138	0.964
			θ	1.008	0.175	0.171	0.941	1.009	0.159	0.152	0.930
			β_1	0.532	0.163	0.157	0.956	0.523	0.146	0.141	0.948
		250	α	1.217	0.119	0.125	0.938	1.216	0.109	0.115	0.948
			θ	1.009	0.135	0.141	0.940	1.006	0.122	0.121	0.944
			β_1	0.522	0.124	0.132	0.943	0.518	0.112	0.117	0.954
0.5	0.8	50	α	1.233	0.271	0.227	0.970	1.226	0.246	0.203	0.972
			θ	1.070	0.341	0.332	0.972	1.061	0.304	0.394	0.974
			β_1	0.780	0.394	0.229	0.952	0.793	0.368	0.222	0.952
		150	α	1.211	0.152	0.135	0.966	1.210	0.140	0.126	0.970
			θ	1.019	0.175	0.159	0.976	1.017	0.159	0.151	0.968
			β_1	0.800	0.239	0.173	0.958	0.809	0.222	0.172	0.950
		250	α	1.206	0.118	0.106	0.970	1.206	0.109	0.095	0.972
			θ	1.007	0.132	0.117	0.984	1.007	0.121	0.108	0.980
			β_1	0.813	0.190	0.149	0.970	0.808	0.174	0.142	0.962

Table 4: Simulation results for the exponential baseline lifetime distribution with scale parameter $\theta = 1$, and different choices of (τ_1, τ_2) and (β_1, β_2) under 3-SSLT.

(τ_1, τ_2)	(β_1, β_2)	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
(0.3,0.6)	(0.8,0.6)	50	θ	1.161	0.377	0.432	0.965	1.155	0.360	0.404	0.970
			β_1	0.768	0.345	0.234	0.950	0.775	0.328	0.222	0.944

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Table 4 – continued from previous page

(τ_1, τ_2)	(β_1, β_2)	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
			β_2	0.646	0.295	0.229	0.952	0.639	0.244	0.221	0.945
		150	θ	1.010	0.174	0.170	0.963	1.025	0.170	0.152	0.964
			β_1	0.802	0.211	0.146	0.966	0.813	0.199	0.148	0.959
			β_2	0.617	0.166	0.141	0.954	0.598	0.136	0.128	0.959
		250	θ	1.032	0.138	0.132	0.972	1.027	0.132	0.125	0.973
			β_1	0.797	0.164	0.140	0.948	0.798	0.152	0.134	0.952
			β_2	0.615	0.129	0.131	0.958	0.614	0.108	0.107	0.952
(0.3,0.8)	(0.8,0.6)	50	θ	1.128	0.358	0.360	0.974	1.118	0.341	0.348	0.972
			β_1	0.783	0.324	0.218	0.955	0.778	0.297	0.208	0.961
			β_2	0.597	0.291	0.239	0.944	0.605	0.232	0.201	0.949
		150	θ	1.039	0.181	0.186	0.968	1.040	0.175	0.184	0.962
			β_1	0.791	0.191	0.156	0.940	0.788	0.176	0.150	0.951
			β_2	0.618	0.175	0.156	0.966	0.611	0.136	0.137	0.954
		250	θ	1.023	0.137	0.131	0.960	1.024	0.132	0.128	0.964
			β_1	0.793	0.149	0.129	0.946	0.792	0.137	0.119	0.944
			β_2	0.615	0.134	0.129	0.964	0.610	0.105	0.102	0.962
(0.3,0.6)	(0.8,0.5)	50	θ	1.111	0.354	0.366	0.962	1.099	0.333	0.332	0.972
			β_1	0.744	0.335	0.218	0.943	0.752	0.312	0.211	0.945
			β_2	0.551	0.255	0.203	0.952	0.540	0.209	0.183	0.966
		150	θ	1.024	0.177	0.163	0.974	1.022	0.170	0.157	0.974
			β_1	0.801	0.213	0.156	0.958	0.803	0.197	0.151	0.955
			β_2	0.524	0.142	0.129	0.968	0.513	0.117	0.108	0.952
		250	θ	1.014	0.135	0.133	0.970	1.009	0.129	0.128	0.972
			β_1	0.797	0.165	0.135	0.960	0.803	0.153	0.129	0.956
			β_2	0.513	0.107	0.107	0.952	0.510	0.090	0.092	0.954
(0.3,0.8)	(0.8,0.5)	50	θ	1.127	0.360	0.402	0.978	1.114	0.342	0.398	0.977

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Table 4 – continued from previous page

(τ_1, τ_2)	(β_1, β_2)	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
			β_1	0.756	0.310	0.225	0.946	0.766	0.291	0.213	0.947
			β_2	0.544	0.263	0.234	0.942	0.536	0.204	0.194	0.954
		150	θ	1.044	0.182	0.196	0.970	1.041	0.175	0.182	0.974
			β_1	0.803	0.193	0.155	0.950	0.799	0.178	0.155	0.953
			β_2	0.502	0.138	0.133	0.956	0.499	0.110	0.107	0.952
		250	θ	1.023	0.136	0.134	0.954	1.020	0.131	0.129	0.962
			β_1	0.799	0.150	0.132	0.956	0.801	0.139	0.123	0.962
			β_2	0.507	0.108	0.102	0.948	0.502	0.085	0.089	0.952

Table 5: Simulation results for the Weibull baseline lifetime distribution with sshape parameter $\alpha = 0.8$ and $\theta = 1$, and different choices of (τ_1, τ_2) and (β_1, β_2) under 3-SSLT.

(τ_1, τ_2)	(β_1, β_2)	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
(0.3,0.6)	(0.8,0.6)	50	α	0.831	0.204	0.173	0.972	0.834	0.188	0.159	0.974
			θ	1.188	0.696	0.714	0.944	1.123	0.560	0.714	0.968
			β_1	0.794	0.492	0.246	0.948	0.812	0.448	0.229	0.948
			β_2	0.657	0.418	0.276	0.938	0.656	0.336	0.253	0.933
		150	α	0.815	0.113	0.107	0.962	0.811	0.104	0.098	0.966
			θ	1.064	0.316	0.292	0.946	1.058	0.283	0.256	0.960
			β_1	0.797	0.294	0.197	0.962	0.807	0.266	0.187	0.950
			β_2	0.646	0.242	0.207	0.940	0.629	0.191	0.178	0.951
		250	α	0.798	0.085	0.080	0.950	0.802	0.079	0.074	0.958
			θ	1.041	0.236	0.232	0.946	1.024	0.207	0.196	0.960
			β_1	0.808	0.235	0.168	0.958	0.810	0.210	0.158	0.956
			β_2	0.621	0.183	0.172	0.946	0.617	0.146	0.142	0.956

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Table 5 – continued from previous page

(τ_1,τ_2)	(β_1,β_2)	n	Parameter	$p_f=0.6$				$p_f=0.8$			
				AE	ASE	SSE	CP	AE	ASE	SSE	CP
(0.3,0.8)	(0.8,0.6)	50	α	0.818	0.198	0.173	0.962	0.812	0.185	0.157	0.966
			θ	1.249	0.742	0.612	0.960	1.199	0.627	0.606	0.966
			β_1	0.758	0.457	0.264	0.936	0.755	0.423	0.250	0.944
			β_2	0.644	0.423	0.274	0.942	0.659	0.334	0.251	0.938
		150	α	0.803	0.110	0.102	0.964	0.808	0.104	0.095	0.958
			θ	1.073	0.316	0.308	0.976	1.048	0.279	0.255	0.968
			β_1	0.802	0.287	0.188	0.958	0.811	0.264	0.176	0.958
			β_2	0.622	0.240	0.208	0.937	0.611	0.182	0.173	0.936
		250	α	0.806	0.086	0.077	0.966	0.808	0.080	0.072	0.976
			θ	1.039	0.229	0.228	0.974	1.024	0.206	0.204	0.976
			β_1	0.803	0.224	0.166	0.952	0.806	0.204	0.158	0.960
			β_2	0.631	0.187	0.174	0.954	0.616	0.143	0.141	0.950
(0.3,0.6)	(0.8,0.5)	50	α	0.833	0.205	0.180	0.972	0.830	0.189	0.169	0.970
			θ	1.247	0.759	0.851	0.966	1.174	0.601	0.640	0.962
			β_1	0.767	0.474	0.260	0.935	0.771	0.425	0.248	0.938
			β_2	0.598	0.381	0.263	0.964	0.590	0.303	0.242	0.960
		150	α	0.804	0.111	0.100	0.973	0.809	0.103	0.092	0.971
			θ	1.089	0.329	0.326	0.962	1.060	0.284	0.226	0.962
			β_1	0.791	0.293	0.206	0.945	0.803	0.265	0.184	0.952
			β_2	0.533	0.201	0.184	0.970	0.516	0.156	0.149	0.934
		250	α	0.801	0.086	0.081	0.954	0.802	0.079	0.076	0.952
			θ	1.048	0.237	0.219	0.956	1.034	0.210	0.199	0.957
			β_1	0.792	0.231	0.170	0.946	0.798	0.206	0.163	0.954
			β_2	0.522	0.152	0.145	0.946	0.518	0.122	0.121	0.952
(0.3,0.8)	(0.8,0.5)	50	α	0.833	0.201	0.169	0.974	0.840	0.186	0.157	0.974
			θ	1.175	0.665	0.664	0.960	1.127	0.546	0.505	0.961
			β_1	0.782	0.467	0.251	0.948	0.794	0.426	0.233	0.954

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Table 5 – continued from previous page

(τ_1,τ_2)	(β_1,β_2)	n	Parameter	$p_f=0.6$ AE	ASE	SSE	CP	$p_f=0.8$ AE	ASE	SSE	CP
			β_2	0.595	0.428	0.276	0.948	0.552	0.274	0.228	0.941
		150	α	0.810	0.111	0.096	0.978	0.806	0.102	0.091	0.966
			θ	1.063	0.309	0.284	0.956	1.071	0.283	0.279	0.964
			β_1	0.808	0.286	0.192	0.952	0.804	0.256	0.190	0.940
			β_2	0.529	0.202	0.188	0.958	0.513	0.148	0.141	0.948
		250	α	0.805	0.085	0.079	0.968	0.804	0.078	0.073	0.964
			θ	1.035	0.227	0.211	0.968	1.033	0.206	0.200	0.976
			β_1	0.808	0.224	0.167	0.970	0.809	0.201	0.155	0.962
			β_2	0.522	0.154	0.149	0.950	0.517	0.111	0.123	0.937

Table 6: Simulation results for the Weibull baseline lifetime distribution with sshape parameter $\alpha = 1.2$ and $\theta = 1$, and different choices of (τ_1, τ_2) and (β_1, β_2) under 3-SSLT.

(τ_1,τ_2)	(β_1,β_2)	n	Parameter	$p_f=0.6$ AE	ASE	SSE	CP	$p_f=0.8$ AE	ASE	SSE	CP
(0.3,0.6)	(0.8,0.6)	50	α	1.242	0.377	0.297	0.964	1.239	0.338	0.269	0.966
			θ	1.268	0.822	0.804	0.972	1.260	0.750	0.743	0.966
			β_1	0.790	0.490	0.262	0.938	0.784	0.442	0.255	0.944
			β_2	0.630	0.340	0.244	0.940	0.628	0.291	0.237	0.960
		150	α	1.223	0.207	0.184	0.964	1.222	0.190	0.169	0.963
			θ	1.087	0.327	0.305	0.976	1.078	0.295	0.301	0.972
			β_1	0.795	0.292	0.199	0.946	0.792	0.262	0.195	0.944
			β_2	0.629	0.197	0.186	0.942	0.630	0.169	0.157	0.952
		250	α	1.217	0.158	0.148	0.954	1.215	0.144	0.138	0.962
			θ	1.031	0.229	0.223	0.970	1.025	0.206	0.207	0.971
			β_1	0.822	0.233	0.168	0.960	0.820	0.210	0.162	0.958

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Table 6 – continued from previous page

(τ_1, τ_2)	(β_1, β_2)	n	Parameter	$p_f = 0.6$ AE	ASE	SSE	CP	$p_f = 0.8$ AE	ASE	SSE	CP
			β_2	0.614	0.149	0.145	0.956	0.610	0.127	0.123	0.959
(0.3,0.8)	(0.8,0.6)	50	α	1.241	0.368	0.266	0.970	1.240	0.339	0.253	0.972
			θ	1.320	0.609	0.604	0.957	1.184	0.602	0.600	0.952
			β_1	0.780	0.482	0.257	0.935	0.778	0.442	0.252	0.938
			β_2	0.624	0.342	0.237	0.942	0.622	0.290	0.227	0.952
		150	α	1.212	0.202	0.178	0.960	1.214	0.189	0.164	0.952
			θ	1.093	0.326	0.337	0.986	1.062	0.284	0.274	0.983
			β_1	0.795	0.288	0.205	0.940	0.810	0.270	0.190	0.948
			β_2	0.616	0.195	0.180	0.940	0.615	0.161	0.153	0.966
		250	α	1.223	0.203	0.166	0.966	1.219	0.189	0.158	0.971
			θ	1.055	0.305	0.287	0.979	1.057	0.283	0.270	0.977
			β_1	0.806	0.291	0.185	0.956	0.800	0.268	0.186	0.958
			β_2	0.630	0.199	0.181	0.960	0.615	0.162	0.150	0.952
(0.3,0.6)	(0.8,0.5)	50	α	1.249	0.374	0.320	0.954	1.247	0.340	0.303	0.948
			θ	1.259	0.683	0.655	0.960	1.217	0.648	0.640	0.968
			β_1	0.772	0.479	0.258	0.946	0.780	0.437	0.252	0.954
			β_2	0.554	0.295	0.258	0.939	0.551	0.253	0.235	0.947
		150	α	1.207	0.202	0.173	0.968	1.203	0.186	0.163	0.969
			θ	1.091	0.330	0.328	0.972	1.067	0.286	0.273	0.974
			β_1	0.795	0.290	0.202	0.956	0.800	0.263	0.192	0.953
			β_2	0.517	0.162	0.156	0.942	0.514	0.139	0.133	0.947
		250	α	1.222	0.158	0.142	0.976	1.220	0.145	0.132	0.974
			θ	1.041	0.232	0.213	0.982	1.023	0.204	0.186	0.972
			β_1	0.819	0.230	0.168	0.964	0.809	0.209	0.155	0.958
			β_2	0.514	0.124	0.122	0.950	0.510	0.105	0.104	0.945
(0.3,0.8)	(0.8,0.5)	50	α	1.229	0.361	0.288	0.960	1.223	0.338	0.281	0.966
			θ	1.248	0.623	0.603	0.942	1.190	0.608	0.597	0.937
			β_1	0.761	0.472	0.260	0.938	0.772	0.442	0.250	0.937

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Table 6 – continued from previous page

(τ_1,τ_2)	(β_1,β_2)	n	Parameter	$p_f=0.6$ AE	ASE	SSE	CP	$p_f=0.8$ AE	ASE	SSE	CP
			β_2	0.554	0.303	0.246	0.942	0.540	0.244	0.215	0.944
		150	α	1.210	0.199	0.170	0.958	1.209	0.186	0.159	0.966
			θ	1.077	0.314	0.281	0.962	1.063	0.285	0.255	0.965
			β_1	0.794	0.286	0.199	0.944	0.798	0.265	0.191	0.942
			β_2	0.514	0.161	0.161	0.958	0.507	0.133	0.129	0.952
		250	α	1.215	0.155	0.127	0.984	1.211	0.146	0.123	0.968
			θ	1.044	0.230	0.207	0.986	1.034	0.209	0.198	0.988
			β_1	0.809	0.228	0.168	0.958	0.809	0.211	0.162	0.957
			β_2	0.509	0.121	0.117	0.952	0.500	0.103	0.102	0.960

Table 7: Simulation results for constrained estimation of the exponential baseline lifetime distribution with scale parameter $\theta = 1$, and $(\tau_1, \tau_2) = (0.3, 0.8)$ and $(\beta_1, \beta_2) = (0.8, 0.5)$ under 3-SSLT.

n	Parameter	$p_f=0.6$ AE	ASE	SSE	CP	$p_f=0.8$ AE	ASE	SSE	CP
	θ	1.097	0.337	0.340	0.962	1.086	0.315	0.323	0.960
50	β_1	0.788	0.189	0.317	0.972	0.792	0.184	0.296	0.968
	β_2	0.498	0.178	0.431	0.928	0.494	0.153	0.355	0.932
	θ	1.032	0.166	0.179	0.959	1.031	0.162	0.172	0.957
150	β_1	0.801	0.145	0.193	0.965	0.798	0.142	0.177	0.965
	β_2	0.504	0.120	0.221	0.937	0.502	0.097	0.184	0.939
	θ	1.028	0.134	0.138	0.946	1.026	0.129	0.132	0.943
250	β_1	0.799	0.129	0.150	0.959	0.799	0.123	0.138	0.956
	β_2	0.501	0.100	0.163	0.942	0.501	0.082	0.133	0.945

5 Data Analysis

This section illustrates the proposed methodology using two real data sets. We first consider a data set from a simple SSLT on lifetime of light-emitting diode (LED) with observations from 60 units, of which 36 are failures and 24 are right-censored. Under normal operating condition, 2 failures and 14 censored observations are observed. The stress is increased at $\tau = 350$ hours after which there are 34 failures and 10 censored observations. Wang et al. (2012) analyzed this data using a TRV modeling and assuming a number of distributions for the baseline lifetime, of which Weibull turned out to be the best fitted model on the basis of the likelihood values. For illustration, we analyze the same data set with TRV modeling and assuming four distributions, namely, exponential with mean θ, Weibull with scale parameter θ and shape parameter α so that mean is $\theta\Gamma(1+\frac{1}{\alpha})$, Log-logistic with scale parameter θ and shape parameter α so that mean is $\dfrac{\theta\pi}{\alpha\,\sin(\pi/\alpha)}$, and Log-normal with location parameter θ and scale parameter α so that mean is $\exp(\theta+\frac{\alpha^2}{2})$, for the baseline lifetime. These four distributions are denoted by Exp(θ), Weib(θ, α), LL (θ, α), and LN(θ, α), respectively. We compare the four distributions by considering their respective Akaike Information Criteria (AIC) values. The results are presented in Table 8. From the AIC values, the Weibull model fits the data best among the four distributions, as in Wang et al. (2012). Also, as expected for LED, the estimated baseline lifetime is found to have increasing failure rate distribution. Note that the estimates under Weibull distribution are somewhat different from those of Wang et al. (2012) possibly due to some convergence issue, as the maximum log-likelihood value is larger (-214.8198) in our analysis compared to the same (-220.2904) of theirs.

In the fish data set from Samanta et al. (2019), swimming performance of 15 fishes are observed against varying water flow rate, starting with an initial flow rate of 15cm/sec, which is increased by 5cm/sec at the pre-fixed time points $110, 130, 150$, and 170 minutes after start, resulting in a 5-SSLT framework. The lifetime is taken as the time when the fish fails to maintain its position in this 5-SSLT experiment. The number of failures at the five stress levels are observed as $4, 6, 0, 3$, and 2, respectively, and there is no censoring. Samanta et al. (2019) have analyzed this data set under CE modeling assuming exponential baseline lifetime distribution with the order restriction that the five successive exponential rate parameters are non-decreasing. Bayesian analysis under the assumption of non-informative prior and order-restricted maximum likelihood estimation of the model parameters have been carried out.

We analyse this data set with and without the isotonic constraints on the tampering coefficients, as discussed in Section 3, under the TRV modeling approach. We assume both exponential and Weibull distributions for baseline lifetime under the initial flow rate. In terms of AIC values, the unconstrained Weibull model seems to give better fit than the exponential model with an increasing failure rate (See Table 9). With proper reparametrization, the maximum likelihood estimates of the model parameters under exponential distribution are found to be exactly the same as the order-restricted estimates in Samanta et al. (2019), as expected in view of the discussion on incorporation of the ordered constraints in Section 3 and the equivalence results of Sultana and Dewanji (2020).

Table 8: Analysis of the LED life test data

Model	MLE of parameters with standard errors in parentheses			AIC
	θ	α	β	
Exp(θ)	9224.41	-	0.018	461.010
	(6.034×10^{-9})	-	(3.099×10^{-3})	
Weib(θ, α)	9224.41	2.392	0.018	435.640
	(1.378×10^{-5})	(0.342)	(0.001)	
LL(θ, α)	9224.41	2.687	0.016	451.779
	(1.772×10^{-5})	(0.393)	(0.002)	
LN(θ, α)	9.976	1.184	0.007	482.277
	(1.348)	(0.354)	(0.009)	

Table 9: Analysis of the Fish data set.

Model	Type of Analysis	MLE of parameters with standard errors in parantheses in parantheses						AIC
		θ	α	β_1	β_2	β_3	β_4	
Exp(θ)	Uncons-trained	396.55	-	0.109	1	0.522	0.718	162.940
		(198.140)	-	(0.074)	(0.518)	(0.405)	(0.655)	
	Constr-ained	171.764	-	1	1	0.262	0.262	171.649
		(51.843)	-	-	-	(0.144)	(0.163)	
Weib(θ, α)	Uncons-trained	396.496	1.326	0.113	1	1	0.577	162.625
		(0.001)	(0.354)	(0.044)	-	(0.485)	(0.462)	
	Constr-ained	142.4591	4.377223	1	1	1	0.962	165.63
		(10.480)	(1.170)	-	-	(0.658)	(0.847)	

6 Concluding Remarks

In this work, we have considered the method of maximum likelihood estimation for the model parameters associated with the baseline lifetime distribution and the tampering coefficients under the multiple step-stress TRV modeling approach with the initial stress condition being the normal one. Similar to the previous estimation work under CE and TFR modeling, the baseline lifetime parameter θ is not estimable when $d_1 = 0$ and, also, there may be some issues if some other d_j's are zero resulting in some estimates being at the limit. The reparametrization used here simplifies the constrained estimation problem to some extent.

The isotonic constraint considered in this work represents the phenomenon that the successive stress levels have more severe impacts. This is an additional feature and we have developed a simulation based likelihood ratio type test for this constraint. In order to illustrate this test through the analysis of the fish data described in Section 5, we perform a pseudo likelihood ratio test for testing the null hypothesis of the isotonic constraints, $H_0 : 1 \geq \beta_1 \geq \cdots \geq \beta_4 \geq 0$, against the alternative hypothesis, H_A : not H_0. This test is based on the statistic T defined as

$$T = 2\{\sup_{H_0 \cup H_A} \log L(\underset{\sim}{\theta}, \underset{\sim}{\beta}) - \sup_{H_0} \log L(\underset{\sim}{\theta}, \underset{\sim}{\beta})\},$$

where $L(\underset{\sim}{\theta}, \underset{\sim}{\beta})$ is the likelihood function of the data given by (3.2). The corresponding p-value is computed using the Monte-Carlo simulation method as described in the following. First on the basis of the given data we calculate the statistic T. Assuming H_0 to be true, we simulate $K = 1000$ data sets of same size n using the estimated model parameters under H_0 and compute the statistic T for each simulated data set. This step gives an estimate of the null distribution of the test statistic T. The p-value is estimated as the proportion of times the value of the statistic T exceeds its observed value (See Geyer (1991)). For the fish data analyzed in Section 5 assuming exponential and Weibull baseline lifetime distribution, the p-values for testing the null hypothesis H_0 turn out to be 0.018 and 0.006, respectively, rejecting the constrained β_i's as in H_0 for both the models. This finding is in line with the results in Table 9 giving lower AIC values for the unconstrained model in both the cases of exponential and Weibull baseline lifetime distributions.

As remarked in Section 3, consistency and asymptotic normality of the maximum likelihood estimators based on random right censored data follow from the standard parametric likelihood theory under some regularity conditions. The estimation procedures of Section 3 clearly follow for the commonly used Type I and Type II censoring plans. The asymptotic results for Type I censored data naturally hold since it is a special case of random censoring. The result of Bhattacharyya (1985) establishes the asymptotic results for Type II censored data also.

As proposed in Sultana and Dewanji (2020), one can model the tampering coefficients, β_j's, as functions of the stress levels and interpret the tampering effects in terms of the stress level. The issue of non-estimability of the baseline parameter(s) when there is no failure in the normal stress level is unfortunate. The very purpose of introducing the step-stress life testing is to circumvent the problem of few failures under normal stress level, which seems to be defeated by this non-estimability issue. Modeling the tampering coefficients as functions of stress level seems to be a way out of this crisis.

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